Sensitivity Analysis with the R^2-Calculus

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Sensitivity Analysis - Review

- Cornfield et al. (1959): Association between smoking and lung cancer; unmeasured confounder must be 9 times as strong in smokers than in non-smokers.

- Rosenbaum (1987): $\pi_i = \mathbb{P}(D_i = 1 \mid Y_i(0), Y_i(1), X_i),\ i \in \{0,\ldots,2n\}$
$$\frac{1}{\Gamma} \leq \frac{\pi_i / (1 - \pi_i)}{\pi_{n+i} / (1 - \pi_{n+i})} \leq \Gamma \quad \text{for } \Gamma \geq 1.$$

- E-values (Ding and VanderWeele 2016), Marginal sensitivity model (Zhao et al. 2019, Dorn and Guo 2022), Instrumental variables (Altonji et al. 2005, Small 2007) and many more

- Plethora of methods but **hardly used in practice**

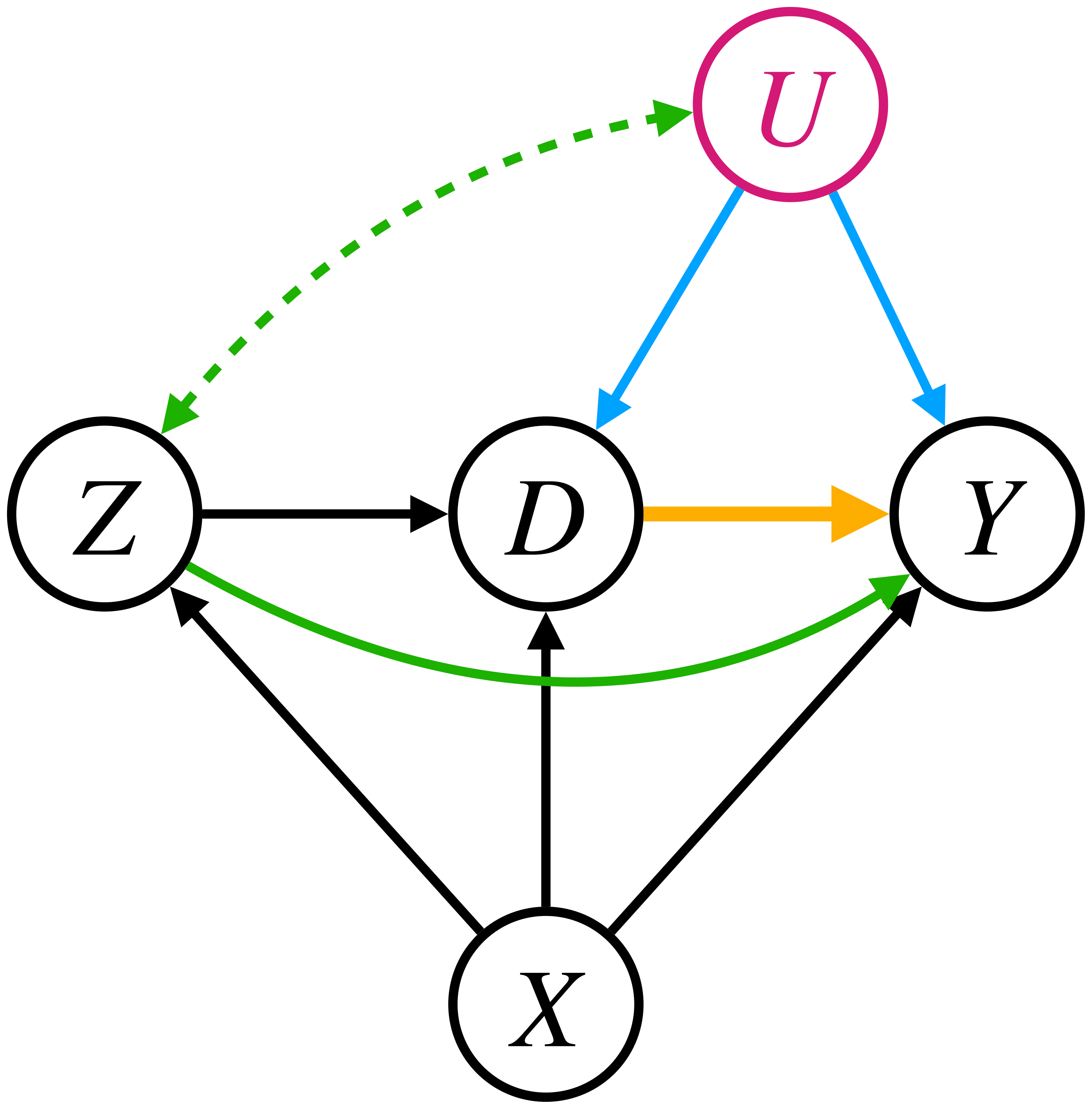
- Reason: Focus on **closed-form solutions** leads to simplistic sensitivity models and unintuitive sensitivity parameters

Sensitivity Analysis - New Framework

- $(V_i, U_i)_{i=1}^n \sim \mathbb{P}_{V,U}$, but only $(V_i)_{i=1}^n$ observed
- Objective: $\beta(\theta, \psi)$
- θ are estimable parameters, i.e. only depend on $\mathbb{P}_V$
- ψ are sensitivity parameters, i.e. depend on $\mathbb{P}_{V,U}$
- Specify domain knowledge as constraints: $g(\theta, \psi) \leq 0, h(\theta, \psi) = 0$

$$\min_{\psi} / \max_{\psi} \beta(\hat{\theta}, \psi) \quad \text{subject to} \quad g(\hat{\theta}, \psi) \leq 0, \; h(\hat{\theta}, \psi) = 0.$$

Linear Model



Objective: $\beta = \beta_{Y\sim D|X,Z,U}$

Ordinary Least Squares (OLS)

$$\beta_{Y\sim D|X,Z} = \frac{\text{cov}(Y^{\perp X,Z}, D^{\perp X,Z})}{\text{var}(D^{\perp X,Z})}$$

Unbiased, if at least one of the arrows is absent.

Two-Stage Least Squares (TSLS)

$$\beta_{\text{TSLS}} = \frac{\text{cov}(Y^{\perp X}, Z^{\perp X})}{\text{cov}(D^{\perp X}, Z^{\perp X})}$$

Unbiased, if both of the arrows are absent.

Parameters θ and ψ : R-values

- Let Y be a random variable; X, W, Z be random vectors.
- Residual of Y after regressing out X denoted by $Y^{\perp X}$

R^2-value : $$R^2_{Y\sim X} := 1 - \frac{\mathrm{var}(Y^{\perp X})}{\mathrm{var}(Y)}$$

Partial R^2-value : $$R^2_{Y\sim X|Z} := \frac{R^2_{Y\sim Z+X} - R^2_{Y\sim Z}}{1 - R^2_{Y\sim Z}}$$

Partial R-value : $$R_{Y\sim X|Z} := \mathrm{corr}(Y^{\perp Z}, X^{\perp Z})$$

Partial f-value : $$f_{Y\sim X|Z} := \frac{R_{Y\sim X|Z}}{\sqrt{1 - R^2_{Y\sim X|Z}}}$$

Cinelli and Hazlett (2020)

R^2-calculus

(i) *Independence*: If $Y \perp\!\!\!\perp X$, then $R^2_{Y\sim X} = 0$.

(ii) *Independent additivity*: If $X \perp\!\!\!\perp W$, then $R^2_{Y\sim X+W} = R^2_{Y\sim X} + R^2_{Y\sim W}$.

(iii) *Decomposition of unexplained variance*: $1 - R^2_{Y\sim X+W} = (1 - R^2_{Y\sim X})(1 - R^2_{Y\sim W|X})$

(iv) *Recursion of partial correlation*:

$$R_{Y\sim X|W} = \frac{R_{Y\sim X} - R_{Y\sim W}R_{X\sim W}}{\sqrt{1 - R^2_{Y\sim W}}\sqrt{1 - R^2_{X\sim W}}}$$

(v) *Reduction of partial correlation*: If X and W is one-dimensional and $Y \perp\!\!\!\perp W$, then

$$R_{Y\sim X|W} = \frac{R_{Y\sim X}}{\sqrt{1 - R^2_{X\sim W}}}$$

(vi) *Three-variable restriction*: If X and W is one-dimensional, then

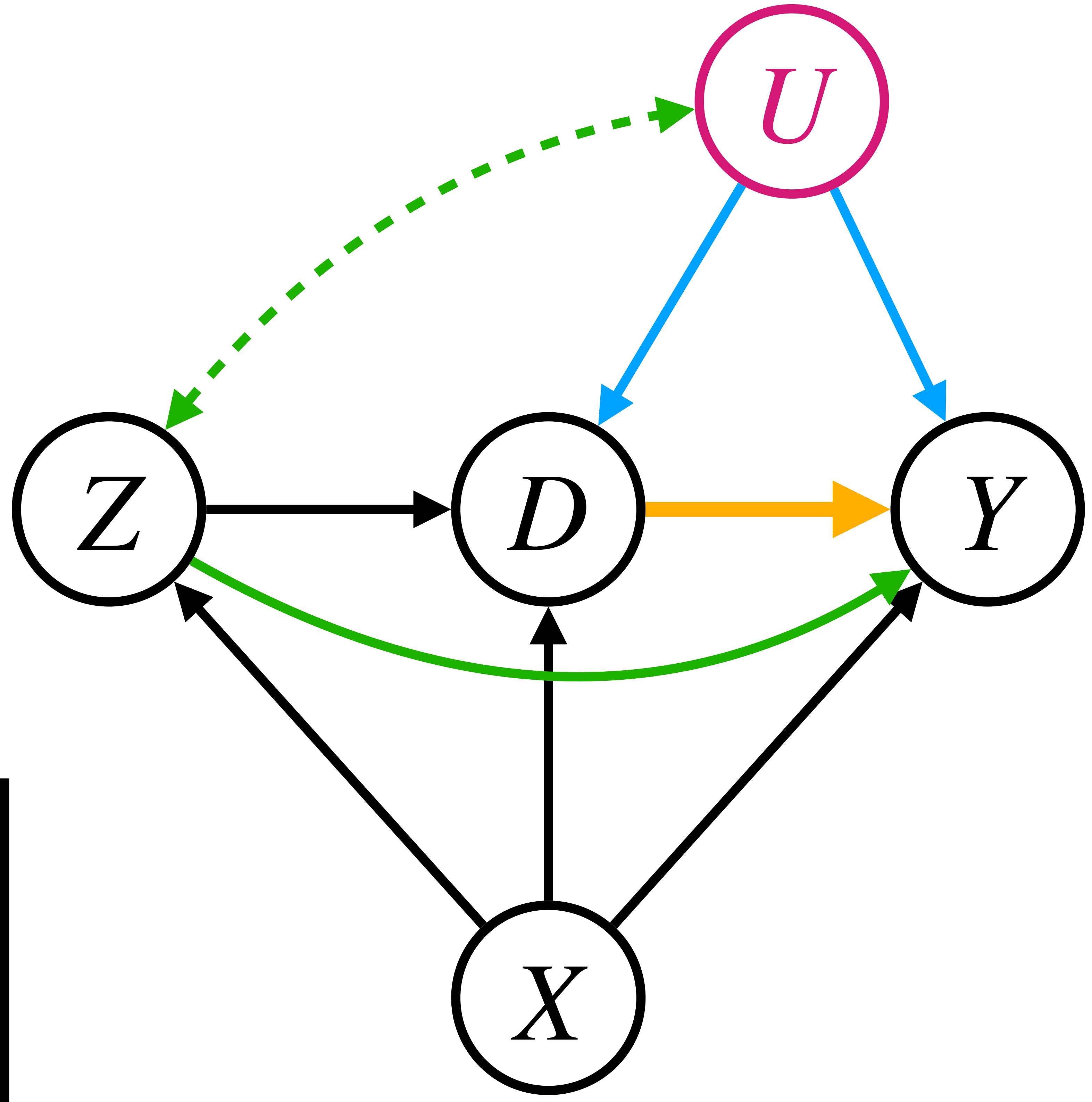
$$f_{Y\sim X|W}\sqrt{1 - R^2_{Y\sim W|X}} = f_{Y\sim X}\sqrt{1 - R^2_{X\sim W}} - R_{Y\sim W|X}R_{X\sim W}.$$

All statements are also true when Z is partialed out, i.e. add "$\,|\,Z$".

Anderson (1958)

Identifying objective β in terms of θ and ψ

- Causal effect: $\beta = \beta_{Y\sim D|X,Z,U}$
- Consider the k-class family of estimands β_k, $k \leq 1$.
- $\beta_1 = \beta_{\text{TSLS}}, \quad \beta_0 = \beta_{Y\sim D|X}, \quad \lim_{k\to-\infty} \beta_k = \beta_{Y\sim D|X,Z}$



$$\beta_k - \beta = \left[\frac{f_{Y\sim Z|X,D}\, R_{D\sim Z|X}}{1 - k + k\, R^2_{D\sim Z|X}} + R_{Y\sim U|X,Z,D}\, f_{D\sim U|X,Z} \right] \frac{\text{sd}(Y^{\perp X,Z,D})}{\text{sd}(D^{\perp X,Z})}$$

Sensitivity parameters ψ : $R_{Y\sim U|X,Z,D}$ and $R_{D\sim U|X,Z}$

Nagar (1959), Theil (1961)

Constraints $g(\theta, \psi) \leq 0$

- Direct bounds, e.g. $R_{D\sim U|X,Z} \in [-0.2, 0.4]$, $R^2_{Y\sim U|X,Z,D} \leq 0.5$
- Assumption: $X = (\dot{X}, \tilde{X})$ such that $U \perp\!\!\!\perp \dot{X} \,|\, \tilde{X}, Z$
- Comparative bound, e.g. $R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z} \leq 2\, R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}$

The unmeasured confounder U can explain at most twice as much variance in Y than $\dot{X}_j$ does - conditional on $(\tilde{X}, \dot{X}_{-j}, Z)$.

Constraints $g(\theta, \psi) \leq 0$ and $h(\theta, \psi) = 0$

- $\min / \max \ \beta_{Y\sim D|X,Z} - R_{Y\sim U|X,Z,D}\, f_{D\sim U|X,Z} \dfrac{\mathrm{sd}(Y^{\perp X,Z,D})}{\mathrm{sd}(D^{\perp X,Z})}$ subject to $(1), (2), (3)$

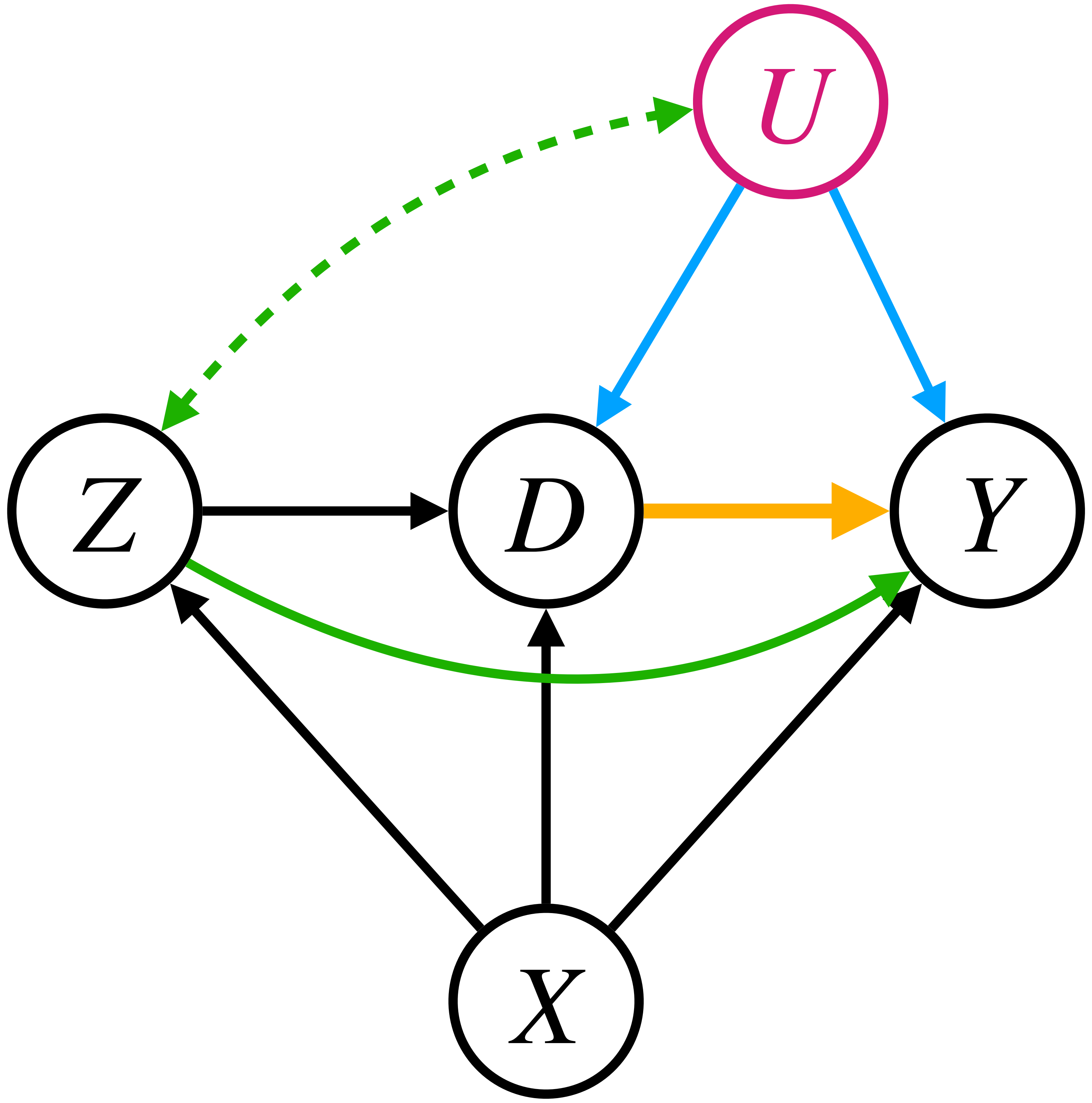
- Bounds: $R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z} \leq 2\, R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}$, $R_{D\sim U|X,Z} \in [-0.2, 0.4]$ $\quad (1)$

- $R^2_{Y\sim U|X,Z} \overset{(v)}{=} \dfrac{R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z}}{1 - R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}}$ $\quad (2)$

- $R_{Y\sim U|X,Z,D} \overset{(iv)}{=} \dfrac{R_{Y\sim U|X,Z} - R_{Y\sim D|X,Z}\, R_{D\sim U|X,Z}}{\sqrt{1 - R^2_{Y\sim D|X,Z}}\sqrt{1 - R^2_{D\sim U|X,Z}}}$ $\quad (3)$

Table of Bounds

$U \rightarrow D$	$R_{D\sim U\mid X,Z} \in [B^l_{UD}, B^u_{UD}]$
	$R^2_{D\sim U\mid \tilde{X},\dot{X}_I,Z} \le b_{UD}\, R^2_{D\sim \dot{X}_J\mid \tilde{X},\dot{X}_I,Z}$
$U \rightarrow Y$	$R_{Y\sim U\mid X,D,Z} \in [B^l_{UY}, B^u_{UY}]$
	$R^2_{Y\sim U\mid \tilde{X},\dot{X}_I,Z} \le b_{UY}\, R^2_{Y\sim \dot{X}_J\mid \tilde{X},\dot{X}_I,Z}$
	$R^2_{Y\sim U\mid \tilde{X},\dot{X}_I,Z,D} \le b_{UY}\, R^2_{Y\sim \dot{X}_J\mid \tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	$R_{Y\sim Z\mid X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Z\sim U\mid \tilde{X},\dot{X}_{-j}} \le b_{UZ}\, R^2_{Z\sim \dot{X}_j\mid \tilde{X},\dot{X}_{-j}}$
$Z \rightarrow Y$	$R_{Y\sim Z\mid X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Y\sim Z\mid X,U,D} \le b_{ZY}\, R^2_{Y\sim \dot{X}_j\mid \tilde{X},\dot{X}_{-j},Z,U,D}$

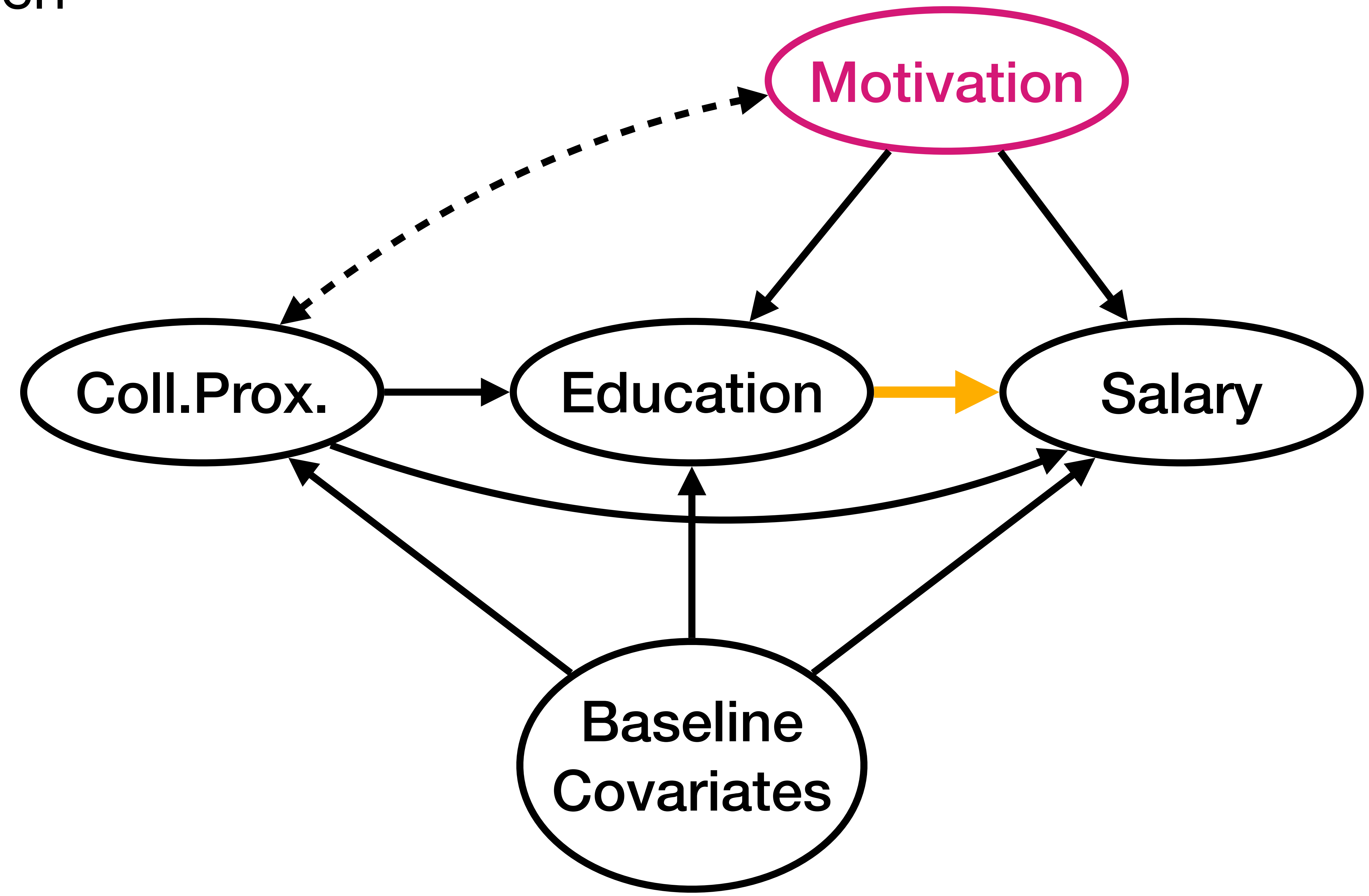


The bounds can be combined in any way.

Cinelli and Hazlett (2020)

Data Example

- National Longitudinal Survey of Young Men
- From 1966 until 1981
- 3010 individuals
- Y: log-earnings
- D: years of schooling
- X: years of labour force experience; indicators for living in the south, being black and living in a metropolitan area
- Z: growing up close to 4-year college



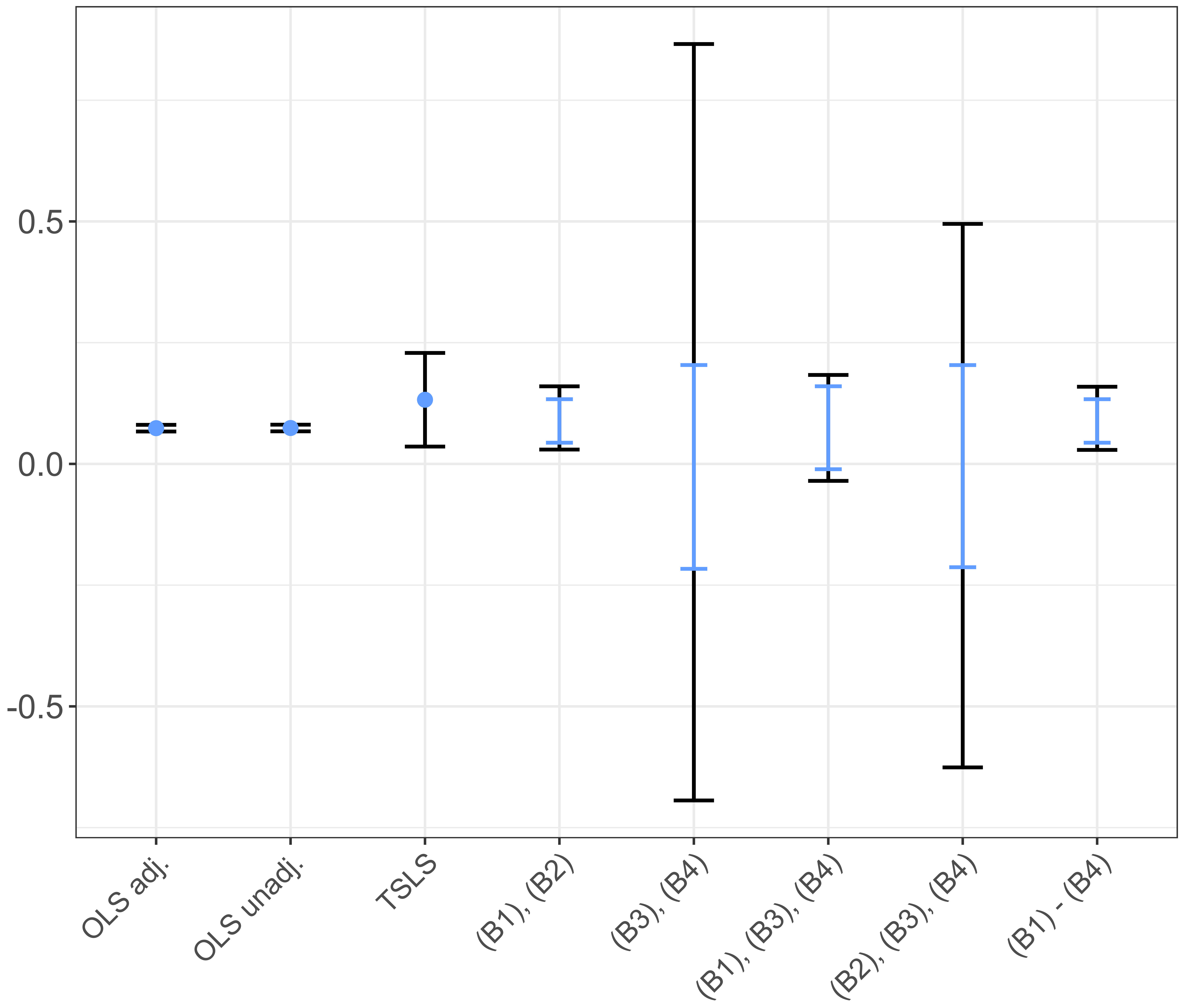
Card (1993)

Sensitivity Analysis

- Partition covariates X: $\dot{X}$ indicator for being black; $\tilde{X}$ remaining covariates
- Assumption: $U \perp\!\!\!\perp \dot{X} \mid \tilde{X}, Z$
- User specified bounds:

(B1) $R^2_{D \sim U | \tilde{X}, Z} \leq 4\, R^2_{D \sim \dot{X} | \tilde{X}, Z}$ (B2) $R^2_{Y \sim U | \tilde{X}, Z, D} \leq 5\, R^2_{Y \sim \dot{X} | \tilde{X}, Z, D}$

(B3) $R^2_{Z \sim U | \tilde{X}} \leq 0.5\, R^2_{Z \sim \dot{X} | \tilde{X}}$ (B4) $R^2_{Y \sim Z | X, U, D} \leq 0.1\, R^2_{Y \sim \dot{X} | \tilde{X}, Z, U, D}$



(B1): $U \rightarrow D$

(B2): $U \rightarrow Y$

(B3): $U \leftrightarrow Z$

(B4): $Z \rightarrow Y$

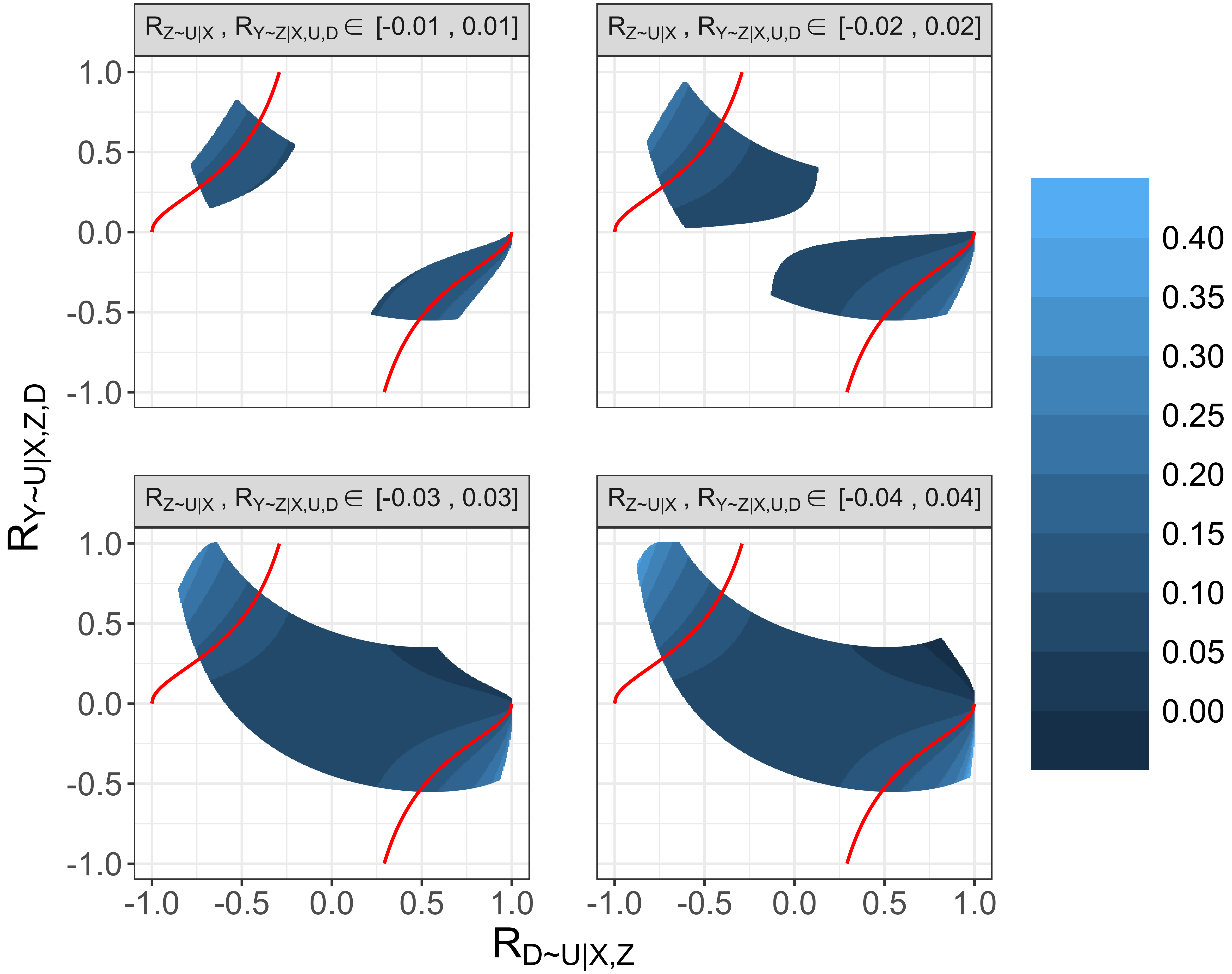
Implicit Constraints Made Explicit

$R_{Z\sim U|X}, R_{Y\sim Z|X,U,D} \in [-r, r]$,

for $r \in \{0.01, 0.02, 0.03, 0.04\}$,

$R^2_{Y\sim U|\tilde{X},Z,D} \leq 5\, R^2_{Y\sim \dot{X}|\tilde{X},Z,D}$

$R_{D\sim U|X,Z} \in [-0.99, 0.99]$



Things I did not talk about

- Extension of the R^2-calculus to Hilbert spaces
- Multiple unmeasured confounders
- Solving the optimization problem
- Confidence statements: Sensitivity intervals via bootstrap
- Additional visualization tools

Conclusion

- New Framework: Sensitivity analysis as optimization problem
- Introduction of the R^2-calculus
- Sensitivity analysis in linear models with R-values (OLS and TSLS)
- Flexible and interpretable bounds
- Application on a dataset

Thanks for your attention!

arXiv: https://arxiv.org/abs/2301.00040

Website: https://tobias-freidling.onrender.com

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Proof of k-class bias

If W is one-dimensional, then

$$\beta_{Y\sim D|X} - \beta_{Y\sim D|X,W} = \frac{\mathrm{cov}(Y^{\perp X}, D^{\perp X})}{\mathrm{var}(D^{\perp X})} - \frac{\mathrm{cov}(Y^{\perp X,W}, D^{\perp X,W})}{\mathrm{var}(D^{\perp X,W})} = R_{Y\sim W|X,D}\, f_{D\sim W|X} \frac{\mathrm{sd}(Y^{\perp X,D})}{\mathrm{sd}(D^{\perp X})}\,.$$

$$\beta_k - \beta = \beta_k - \beta_{Y\sim D|X,Z} + \beta_{Y\sim D|X,Z} - \beta_{Y\sim D|X,Z,U}$$

$$= \ldots = \frac{1}{1-k\left(1-R^2_{D\sim Z|X}\right)}\left(\beta_{Y\sim D|X} - \beta_{Y\sim D|X,Z}\right) + \left(\beta_{Y\sim D|X,Z} - \beta_{Y\sim D|X,Z,U}\right)$$

$$= \ldots = \left[\frac{f_{Y\sim Z|X,D}\, R_{D\sim Z|X}}{1-k+k\,R^2_{D\sim Z|X}} + R_{Y\sim U|X,Z,D}\, f_{D\sim U|X,Z}\right]\frac{\mathrm{sd}(Y^{\perp X,Z,D})}{\mathrm{sd}(D^{\perp X,Z})}$$

Cochran (1938), Frank (2000), Hosman et al. (2010)