Selective Randomization Inference for Adaptive Experiments

Tobias Freidling

Statistical Laboratory, DPMMS
University of Cambridge

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Collaborators

Qingyuan Zhao

Statistical Laboratory, DPMMS
University of Cambridge

Zijun Gao

Marshall School of Business
University of Southern California

Randomization Inference — Standard Set-up

Fisher (1935), Pitman (1937), Zhang & Zhao (2023)

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- P-value: $\mathbb{P}(\ T(Z^*, Y(\cdot), X)\ \leq\ T(Z, Y(\cdot), X)\ \mid Y(\cdot), X, Z),$

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Example

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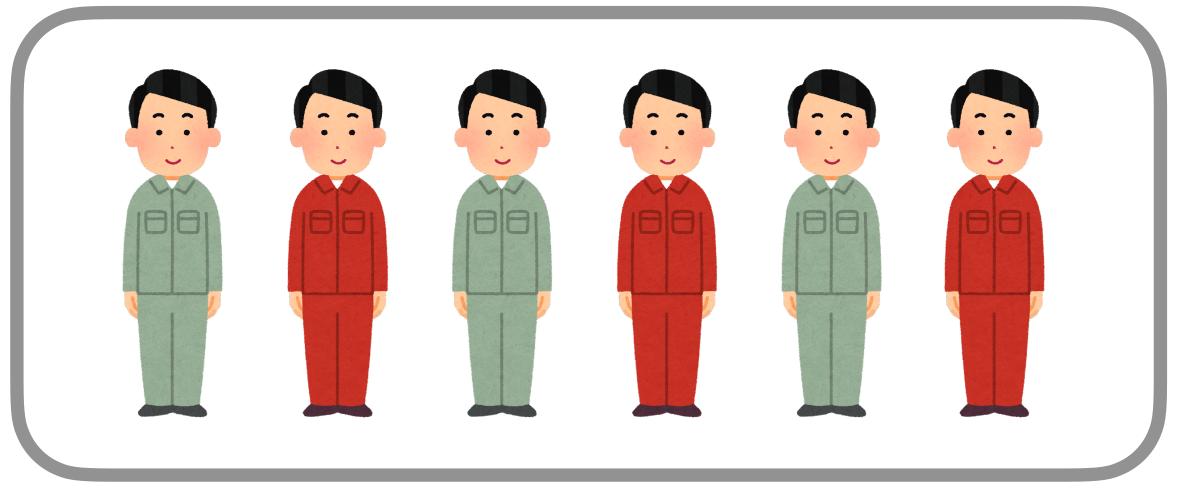

High genetic risk

Low genetic risk

FOURIER trial: Marston et al. (2020)

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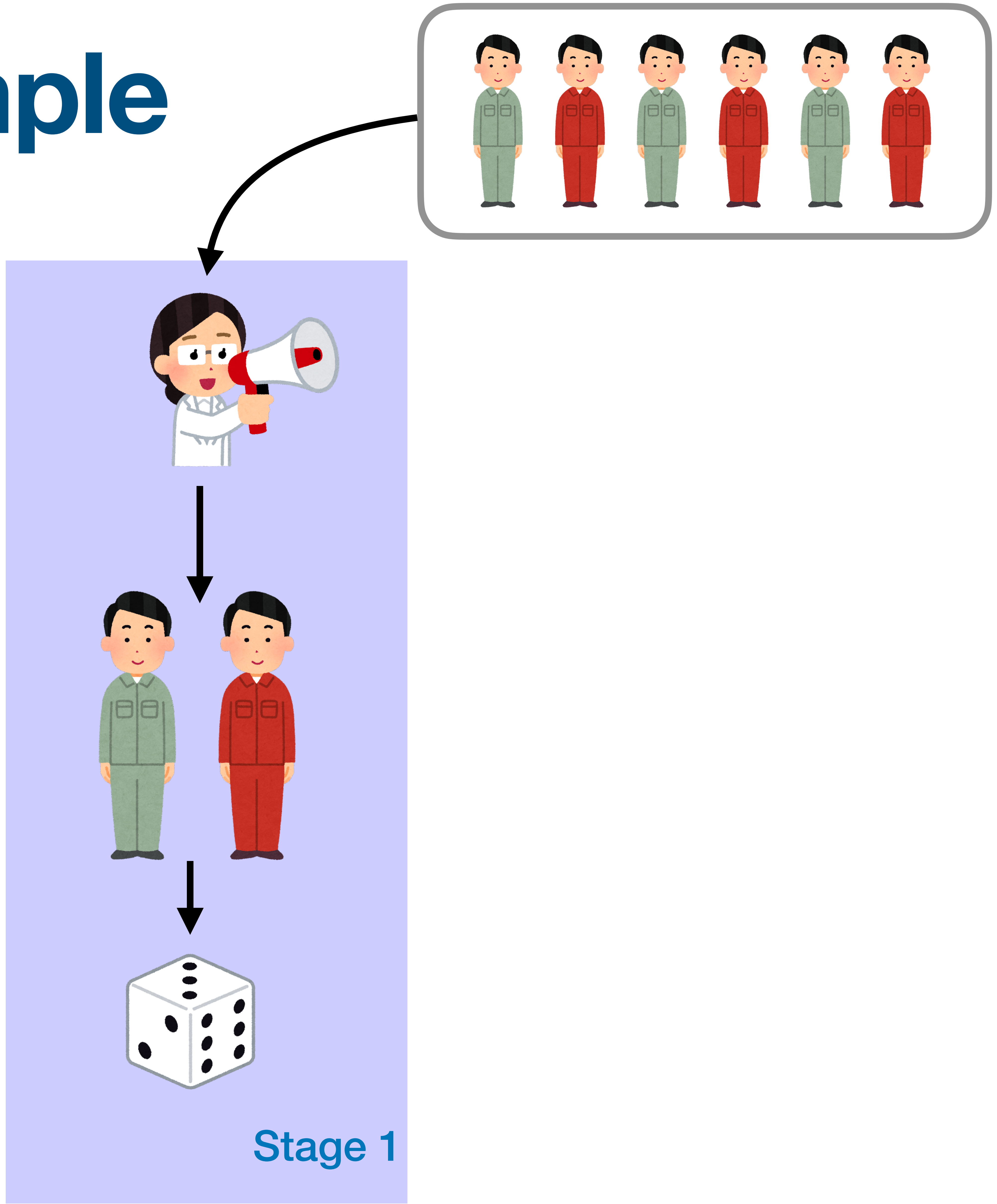


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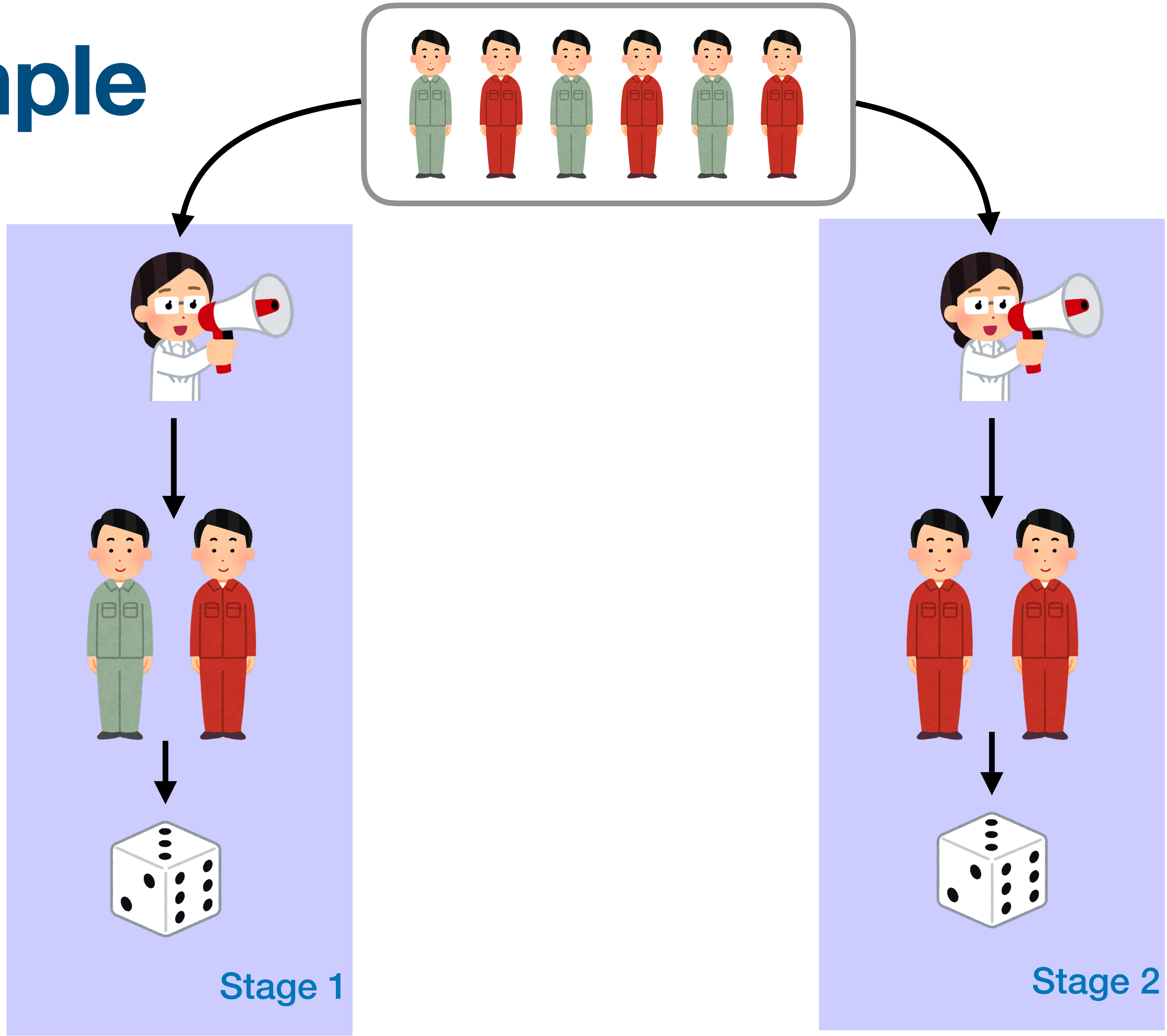

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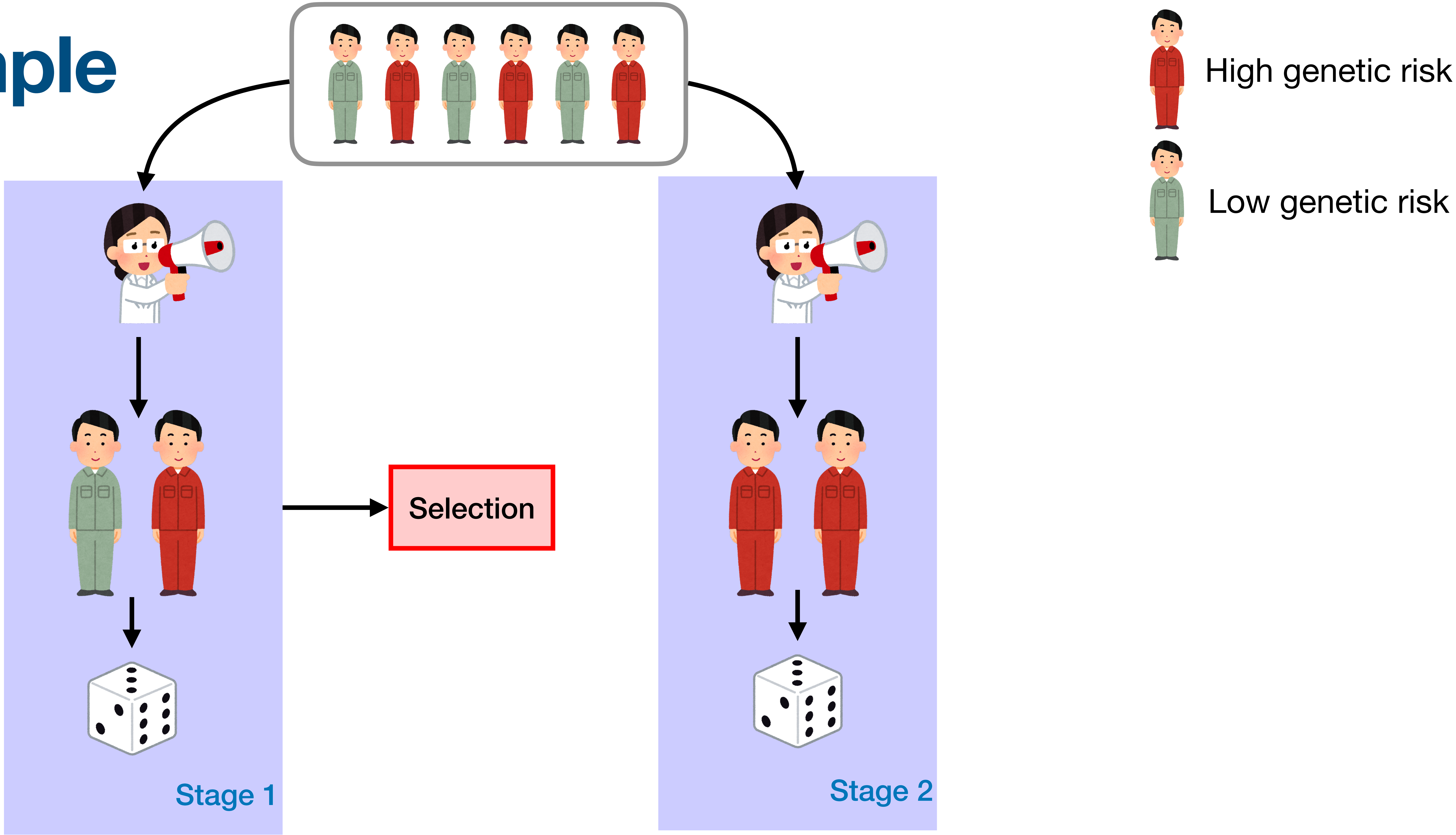

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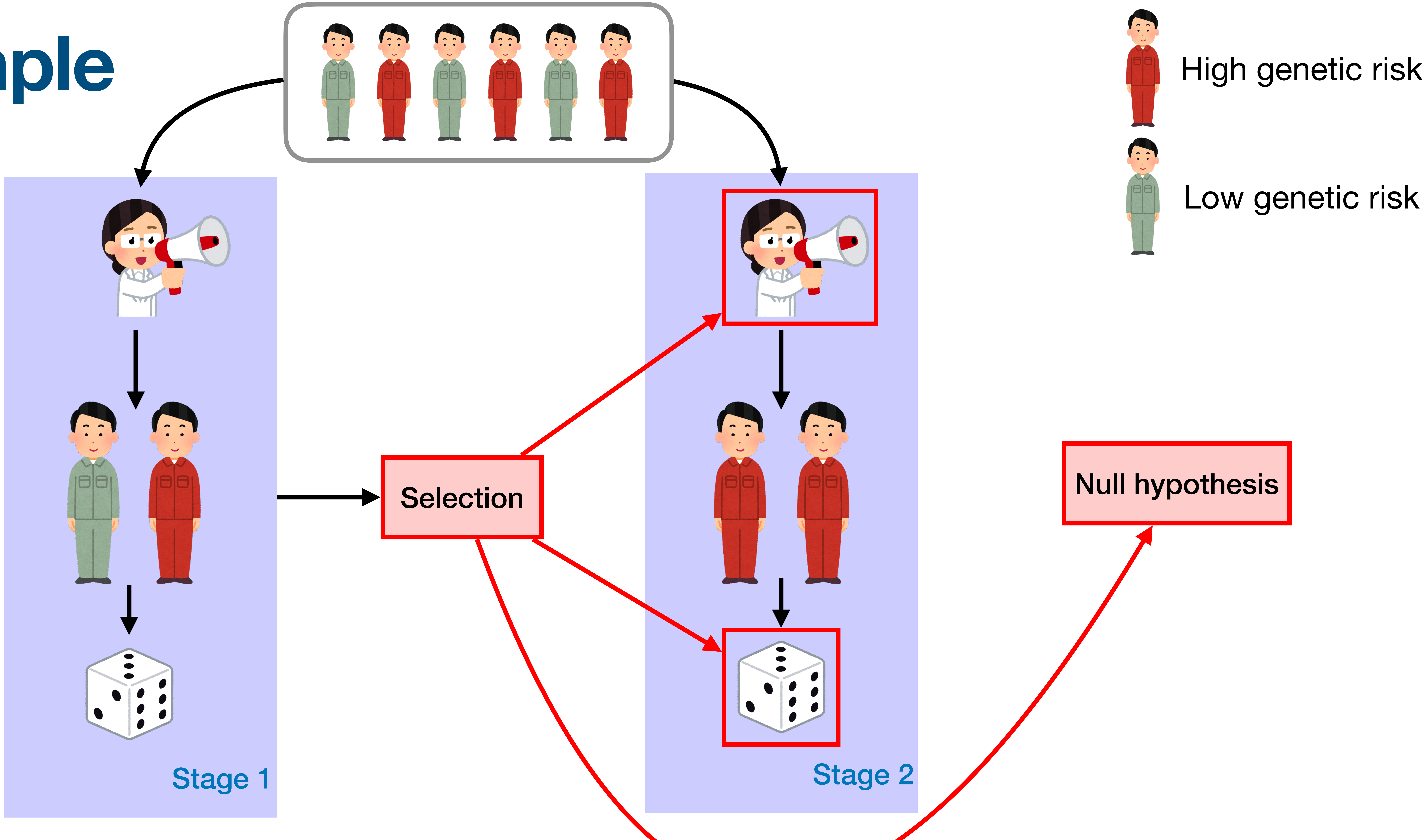

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Graphical Model

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- Covariates: X
- Potential outcomes: $Y(\,\cdot\,)$

$$\boxed{X, Y(\,\cdot\,)}$$

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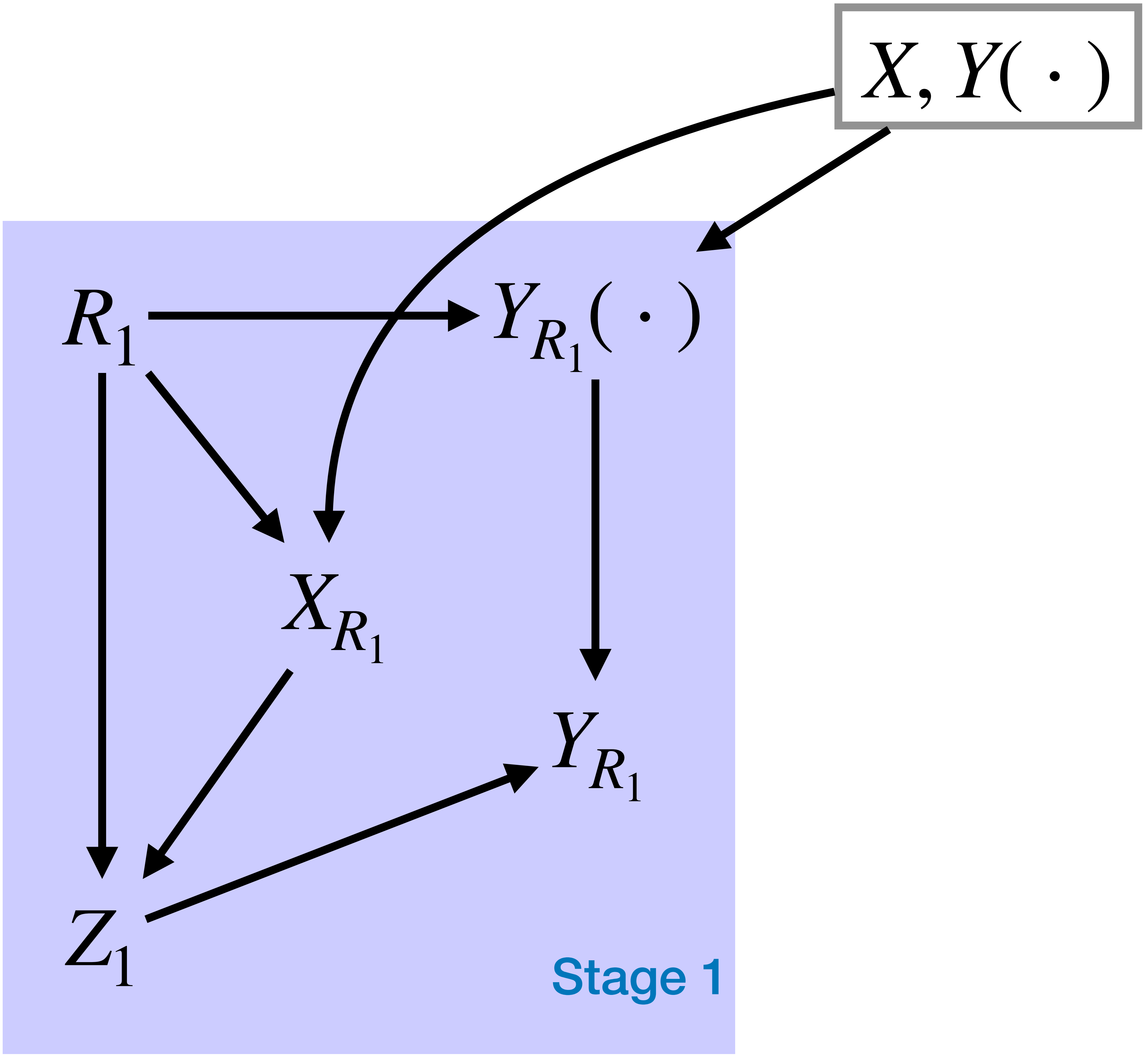

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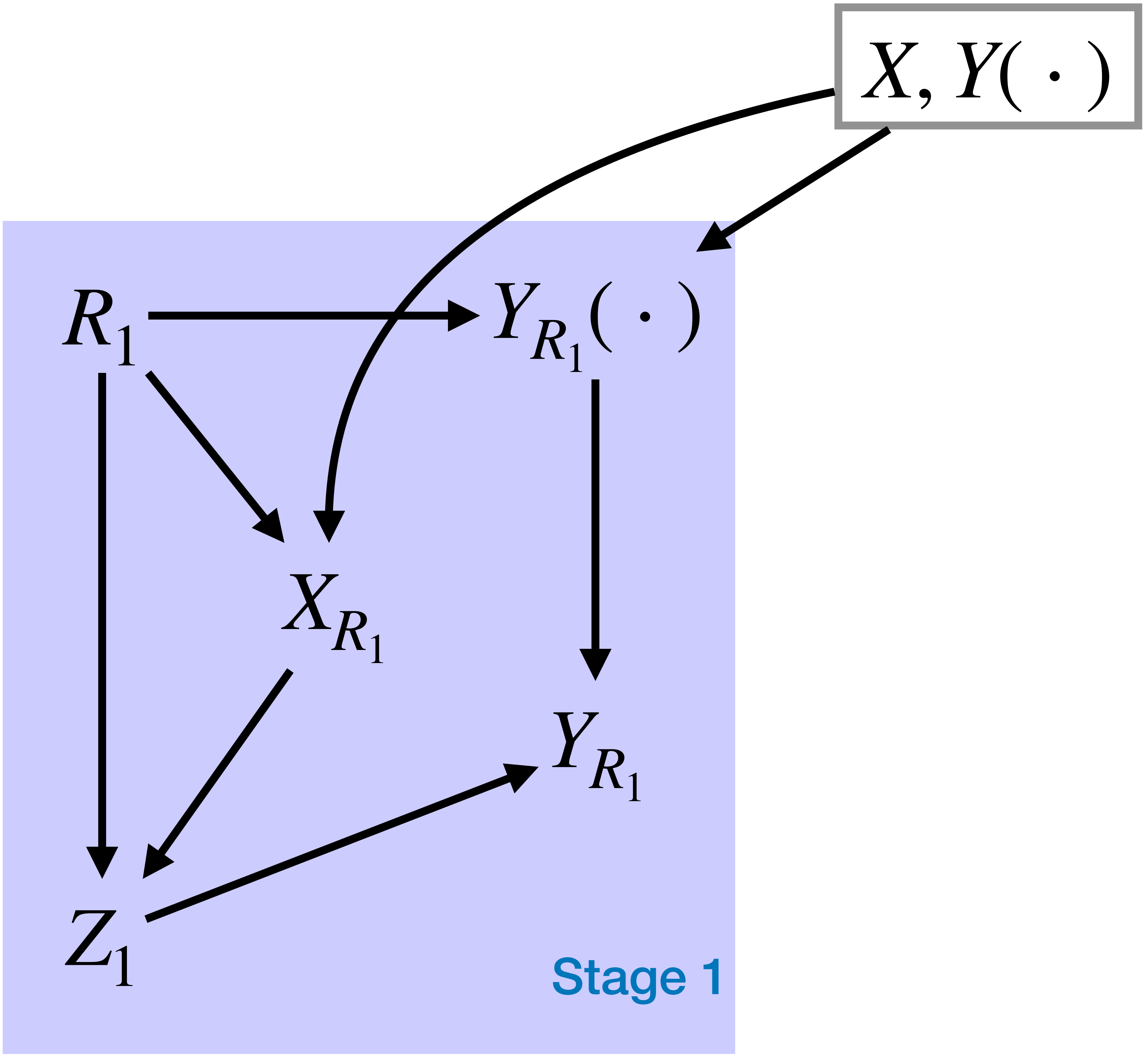

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- Treatments: Z_k
- Observed outcomes: $Y = Y(Z)$

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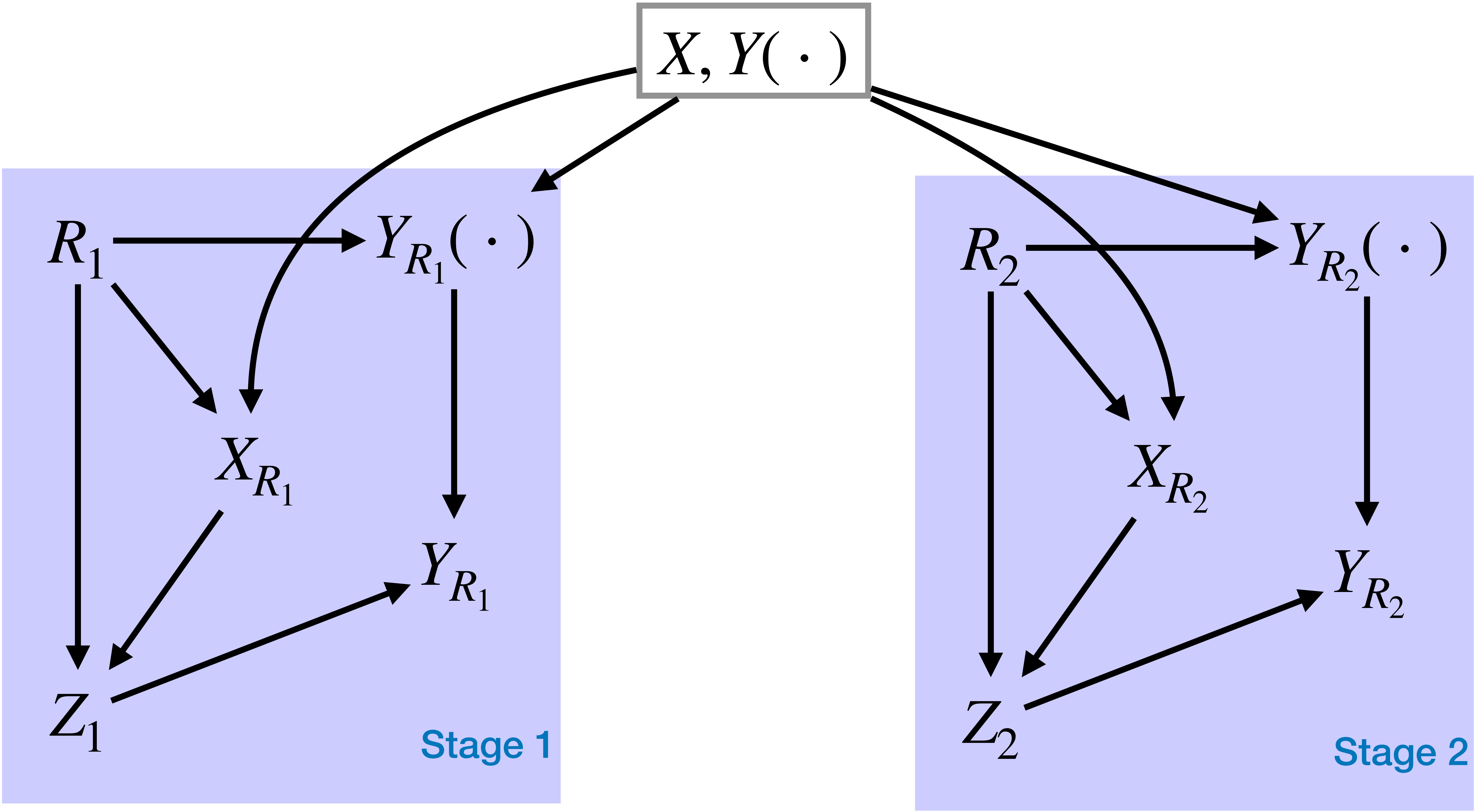

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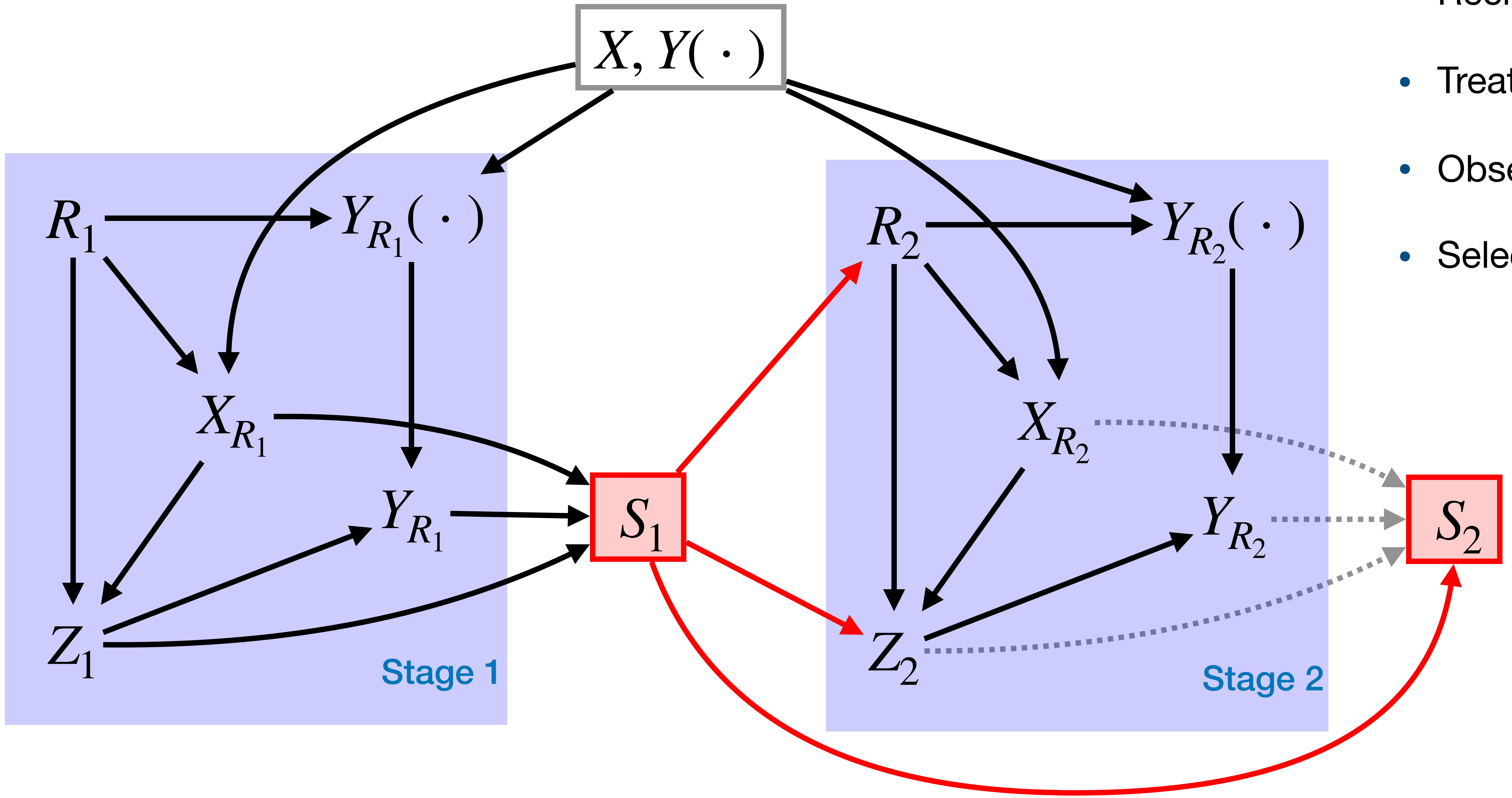

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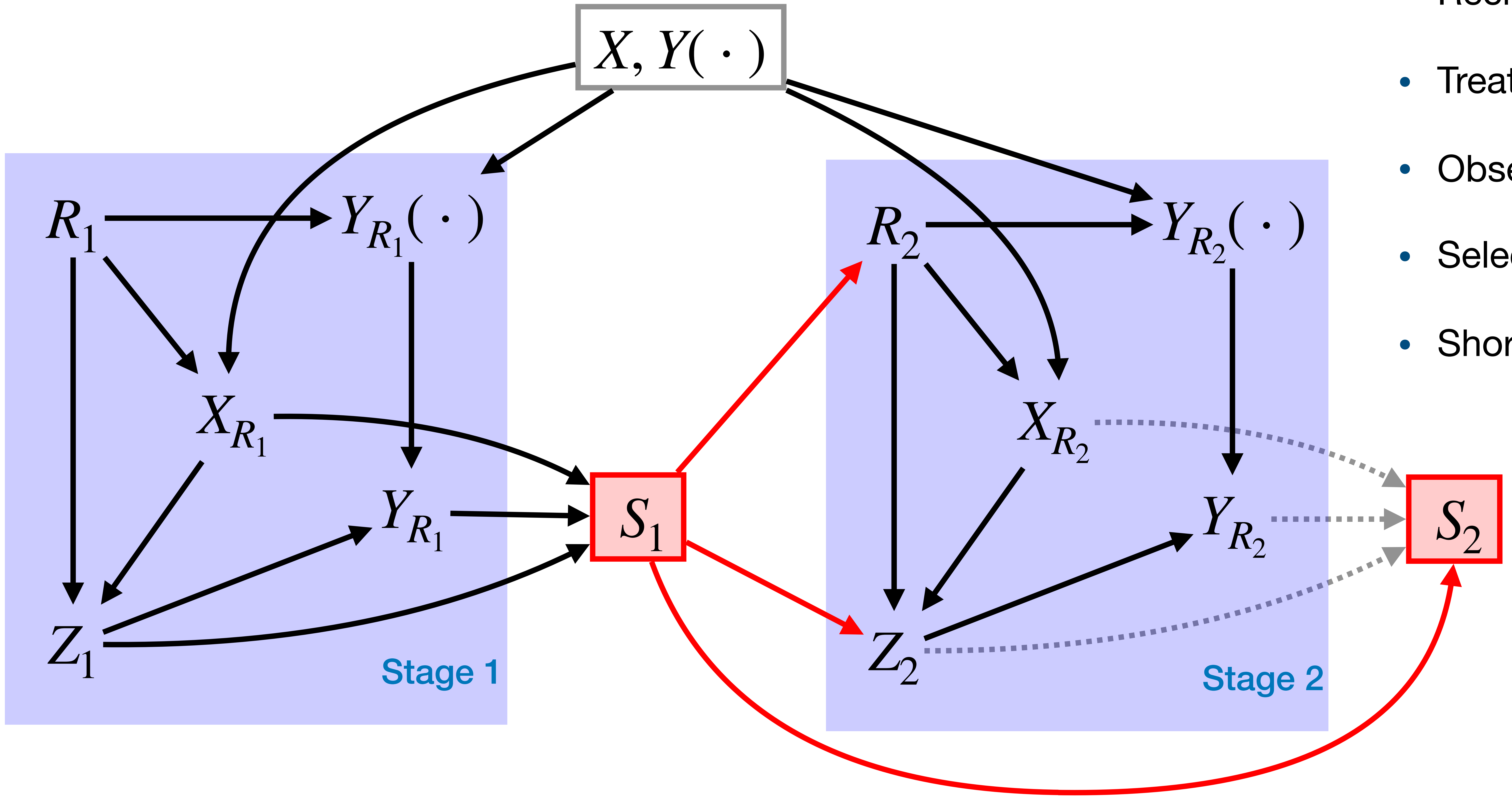

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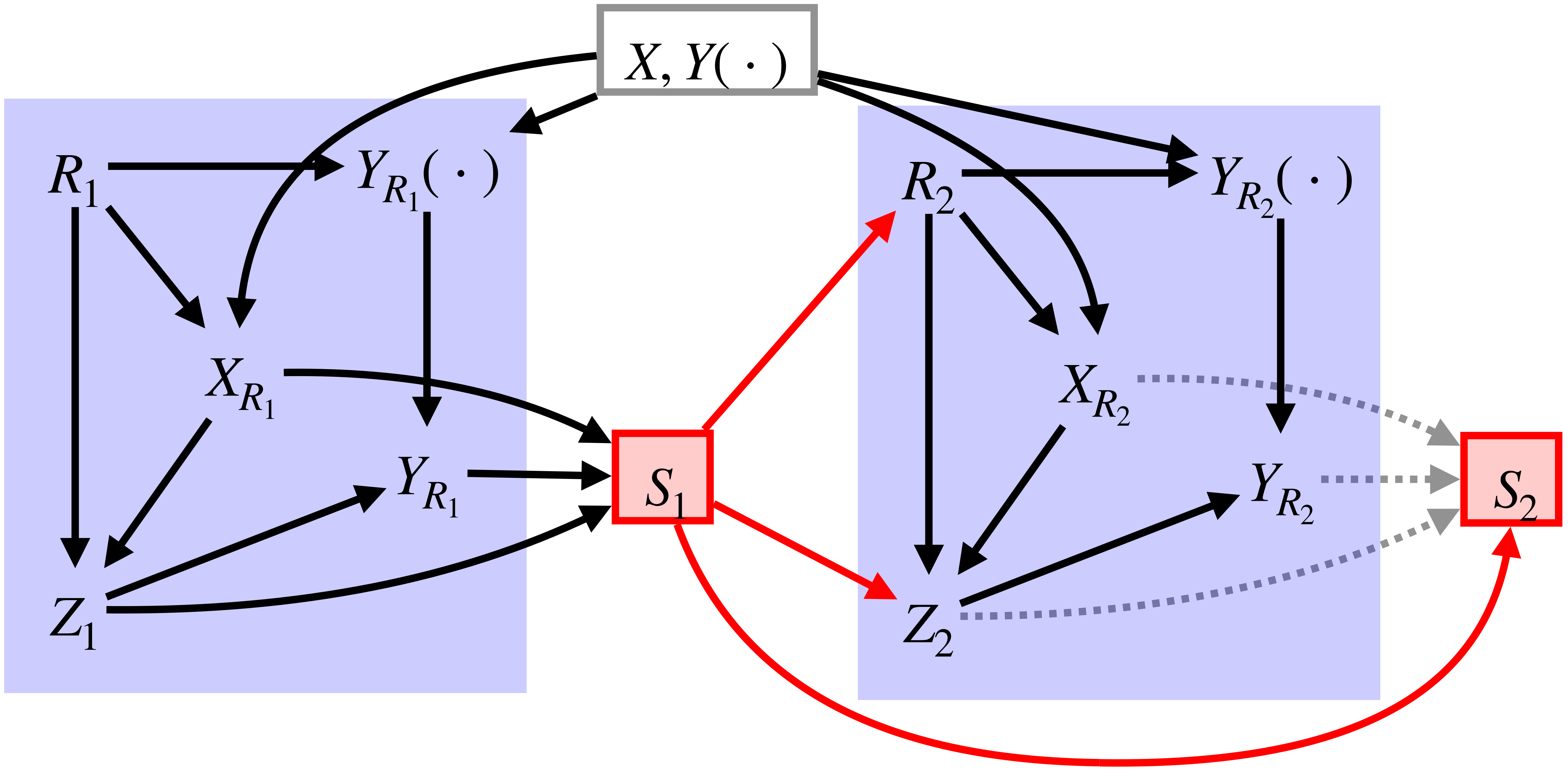
- Short-hand: $W = (R, X_R, Y_R(\cdot))$



$X, Y(\cdot)$
R_1
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X_{R_1}
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Z_1
S_1
R_2
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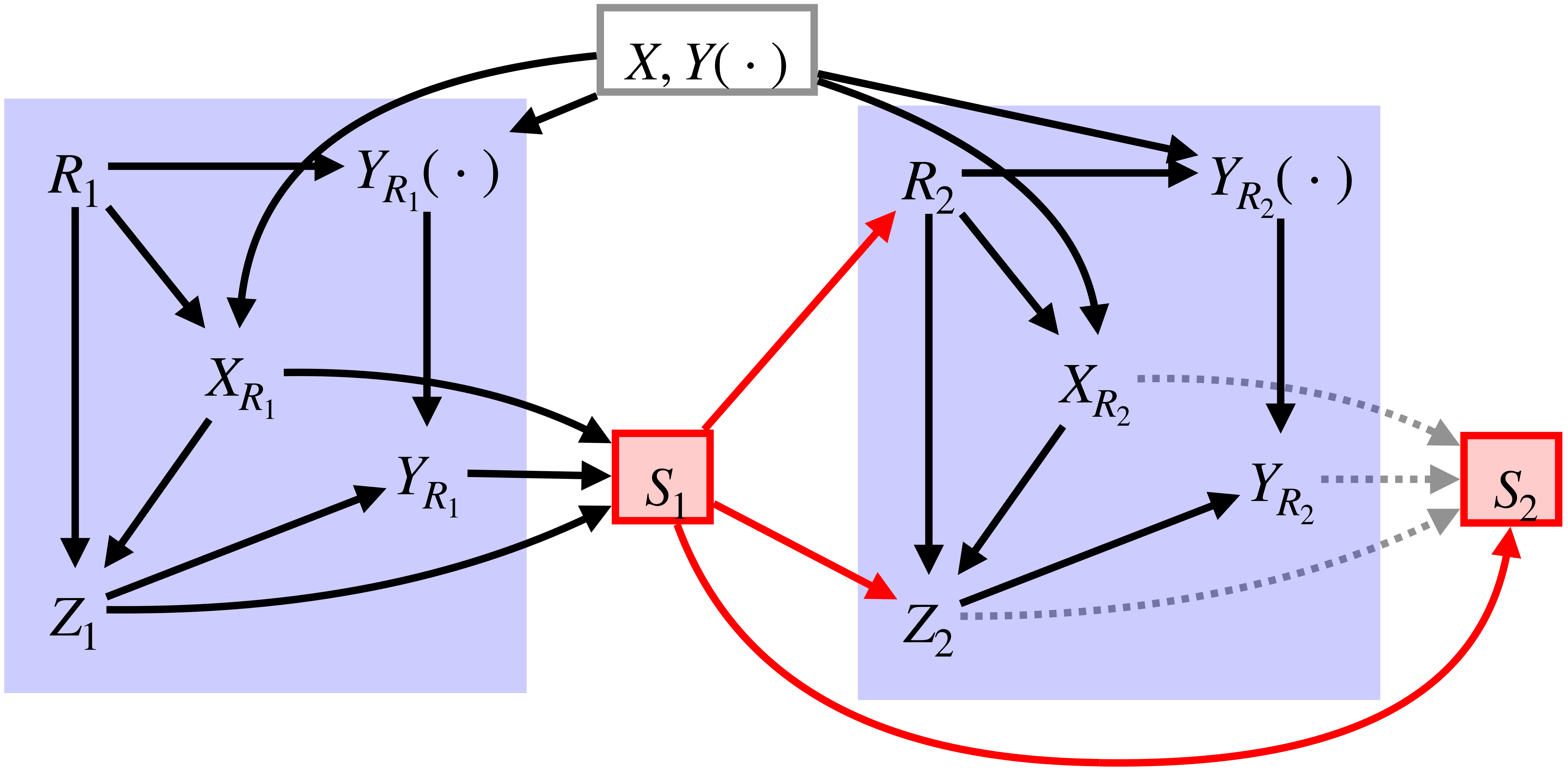

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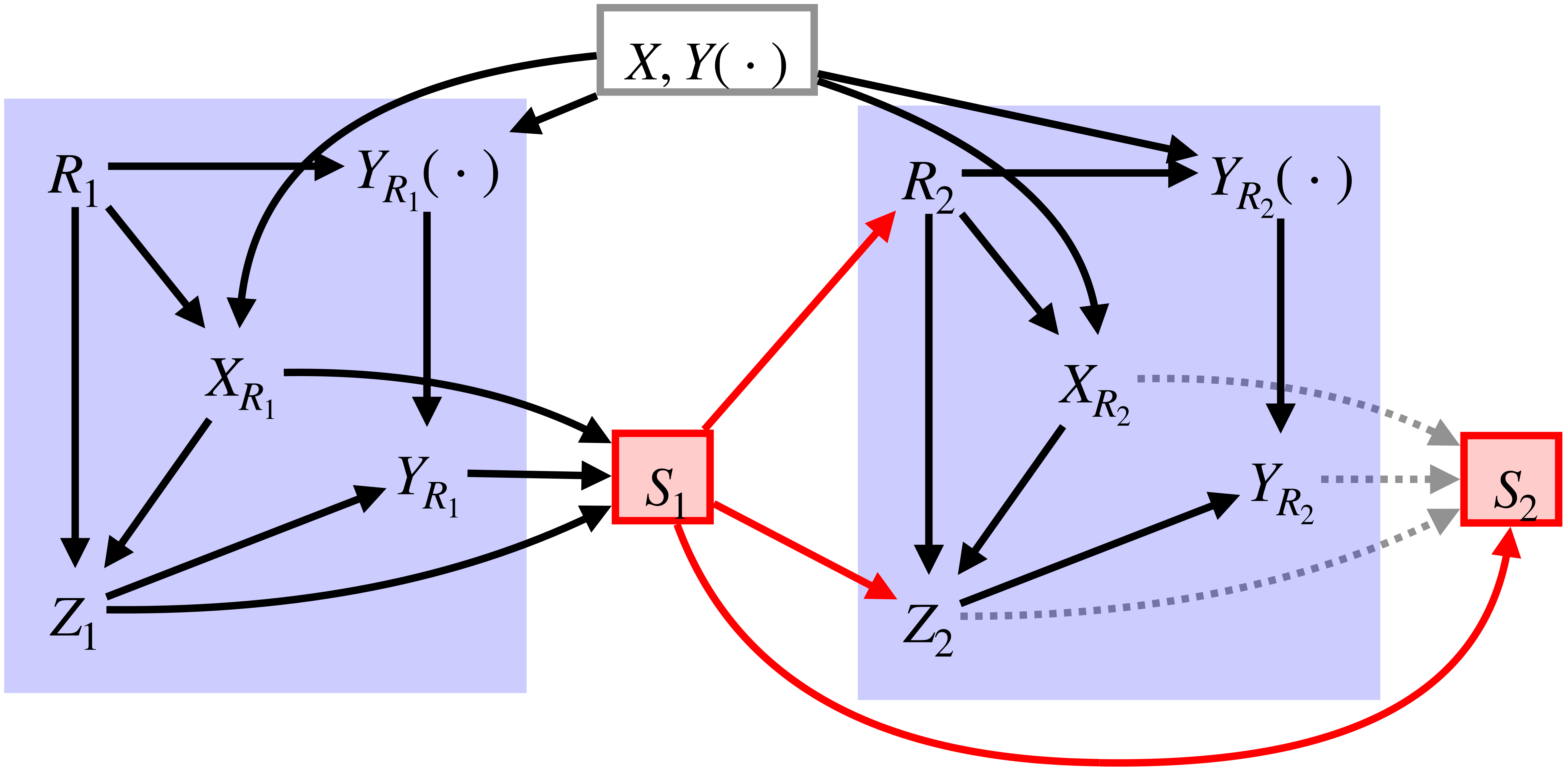

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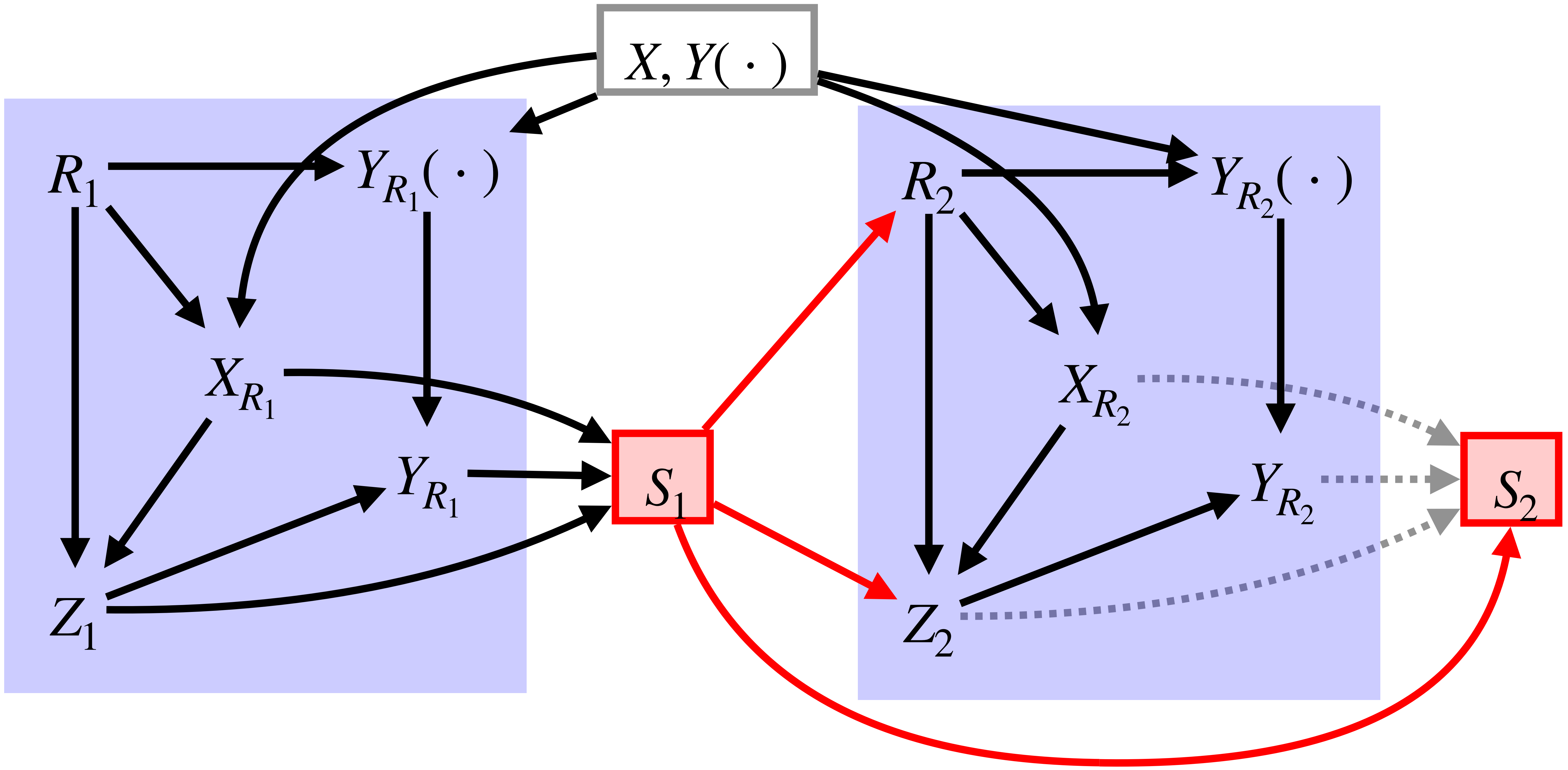

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- Is there a problem when the experiment is adaptive?

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  - Selective randomization inference:

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- Formula for the selective randomization distribution:

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- Rejection sampling, Markov Chain Monte Carlo (MCMC)

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- Data carving: non-adaptive hold-out units

Simulation Study

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- 2 stages, 2 treatments $Z_i \in \{0,1\}$, 2 groups $X_i \in \{\text{low, high}\}$
- Potential outcomes: $Y_i(0) = Y_i(1) \sim N(0,1)$ i.i.d.
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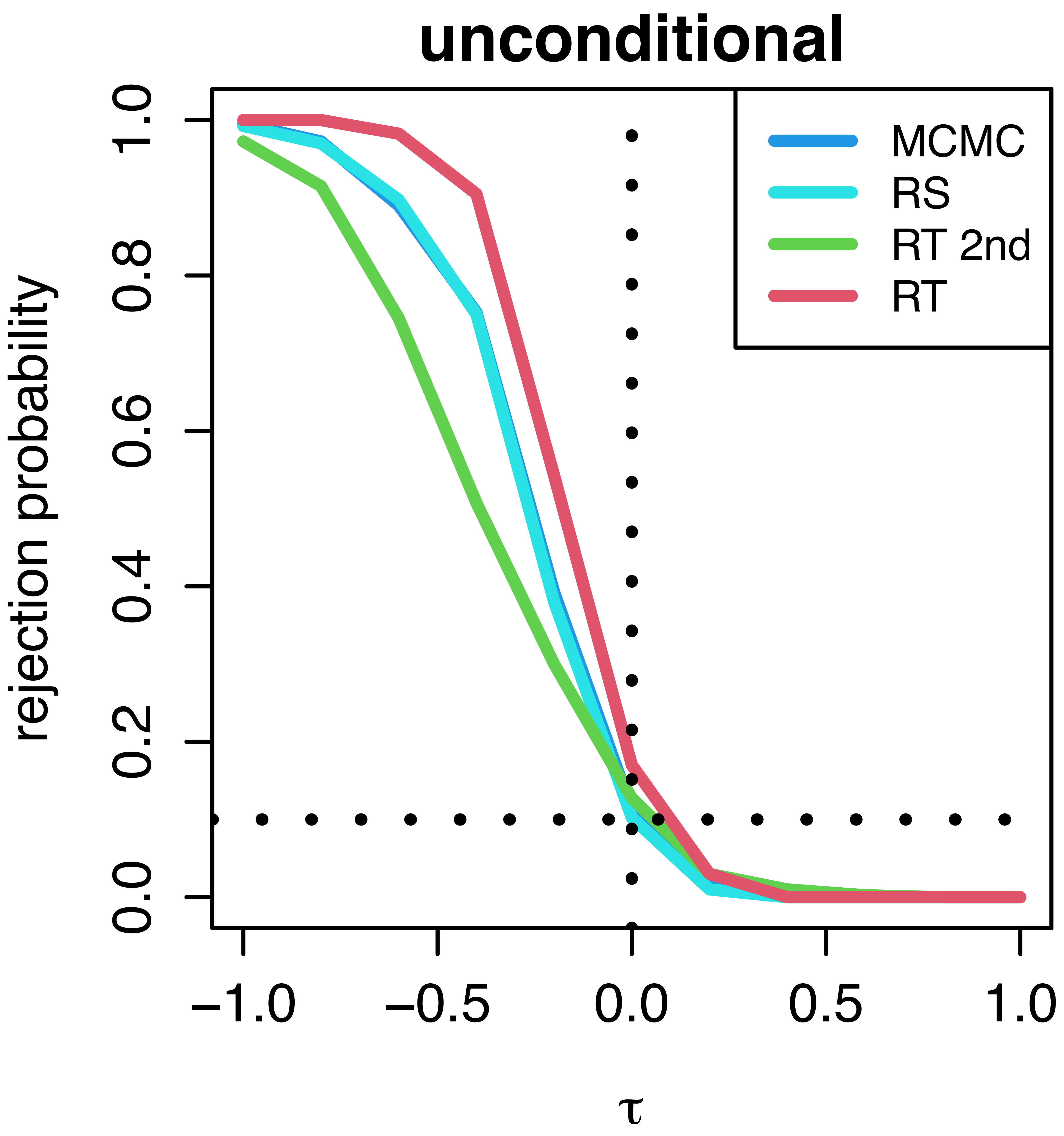
- $\Delta =$ standardized difference in SATEs between groups
- Selection variable:

$$S = \begin{cases} \text{only low,} & \Delta < \Phi^{-1}(0.2), & \text{recruit 40 from group } X_i = \text{low} \\ \text{only high,} & \Delta > \Phi^{-1}(0.8), & \text{recruit 40 from group } X_i = \text{high} \\ \text{both,} & \text{otherwise,} & \text{recruit 20 from each group} \end{cases}$$

Power Analysis



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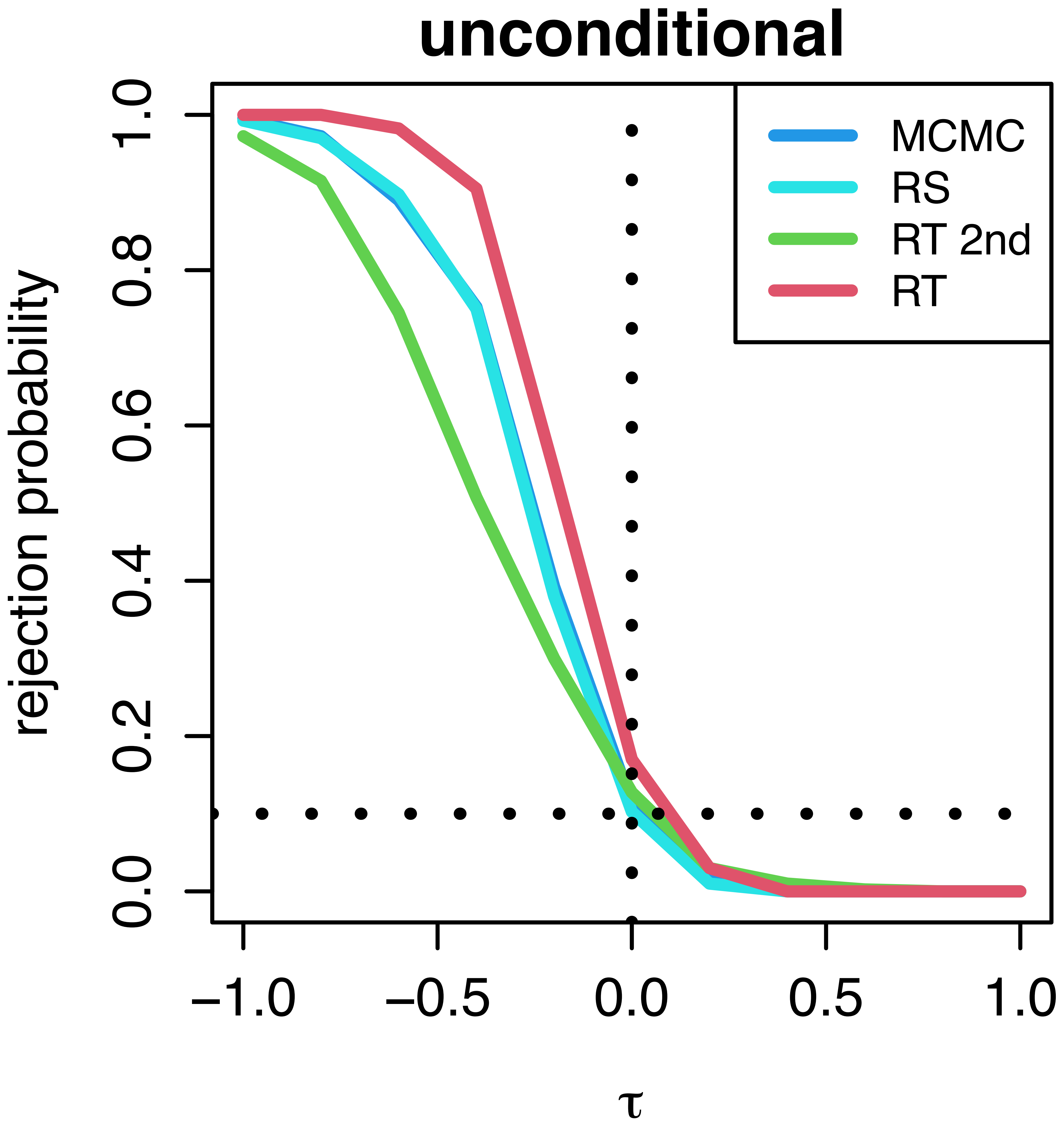

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- RT 2nd: valid but has **low power**

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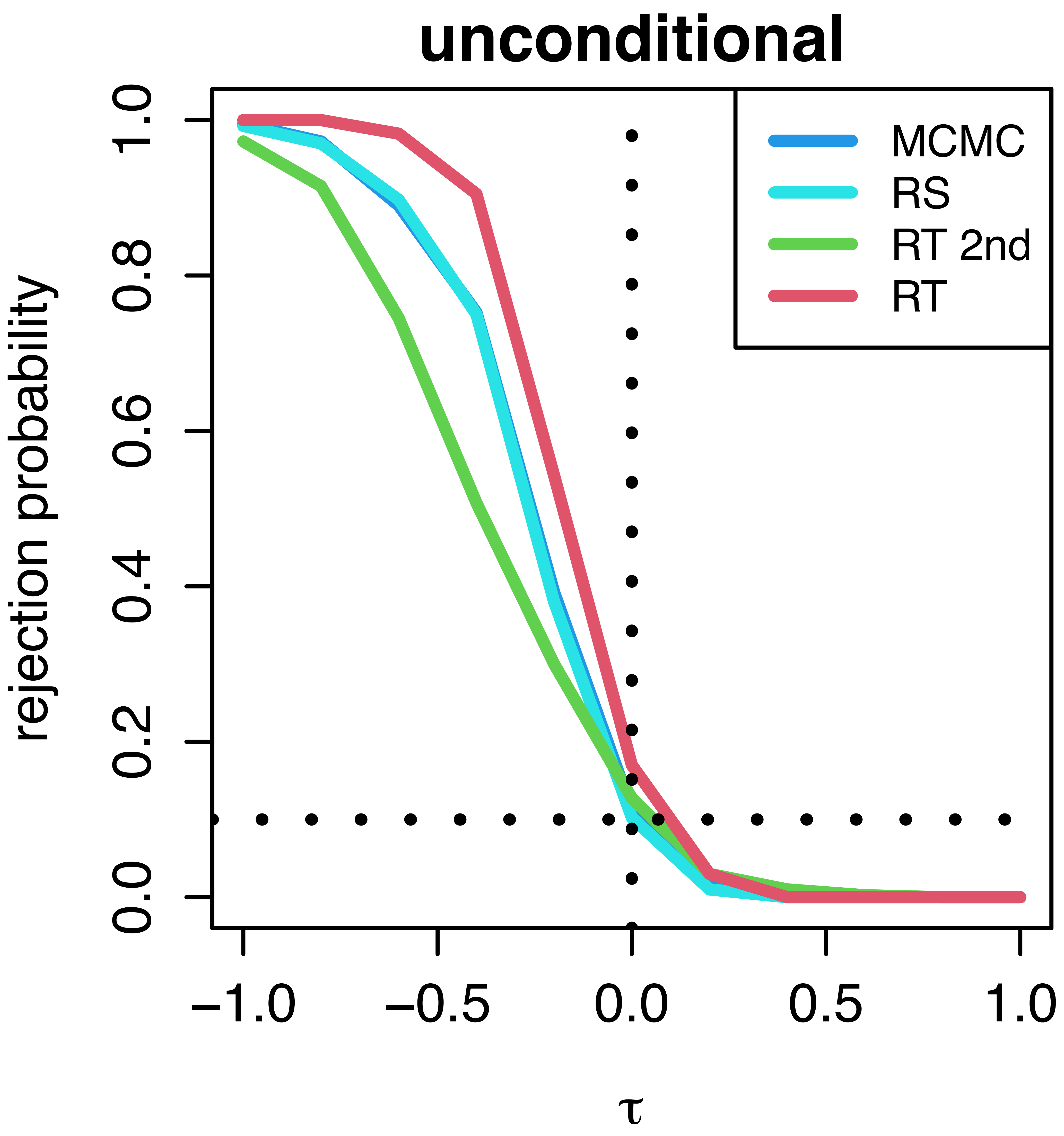

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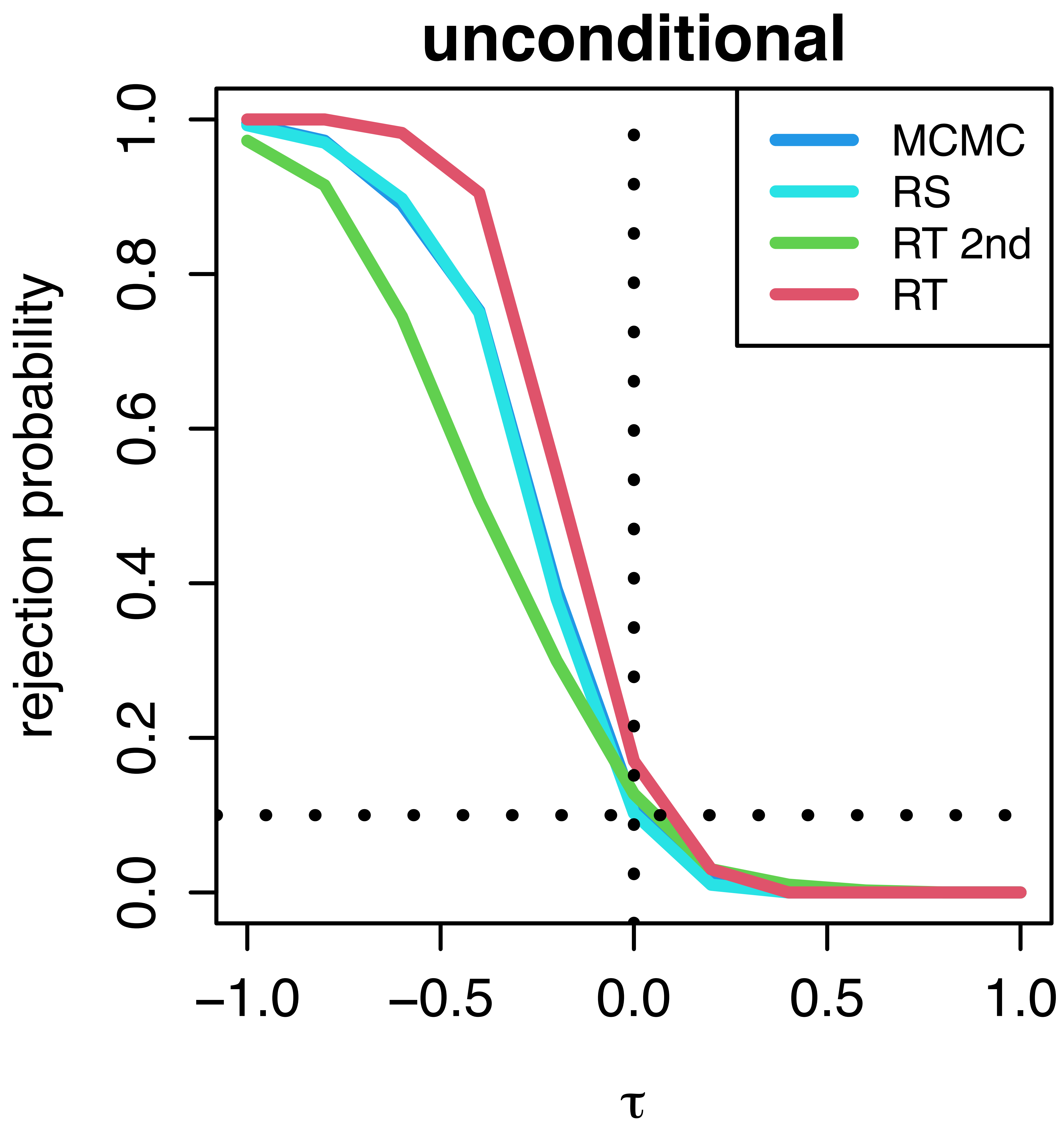

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- Selective RT: **valid and more powerful.**
- Rejection sampling and MCMC lead to very similar approximations.

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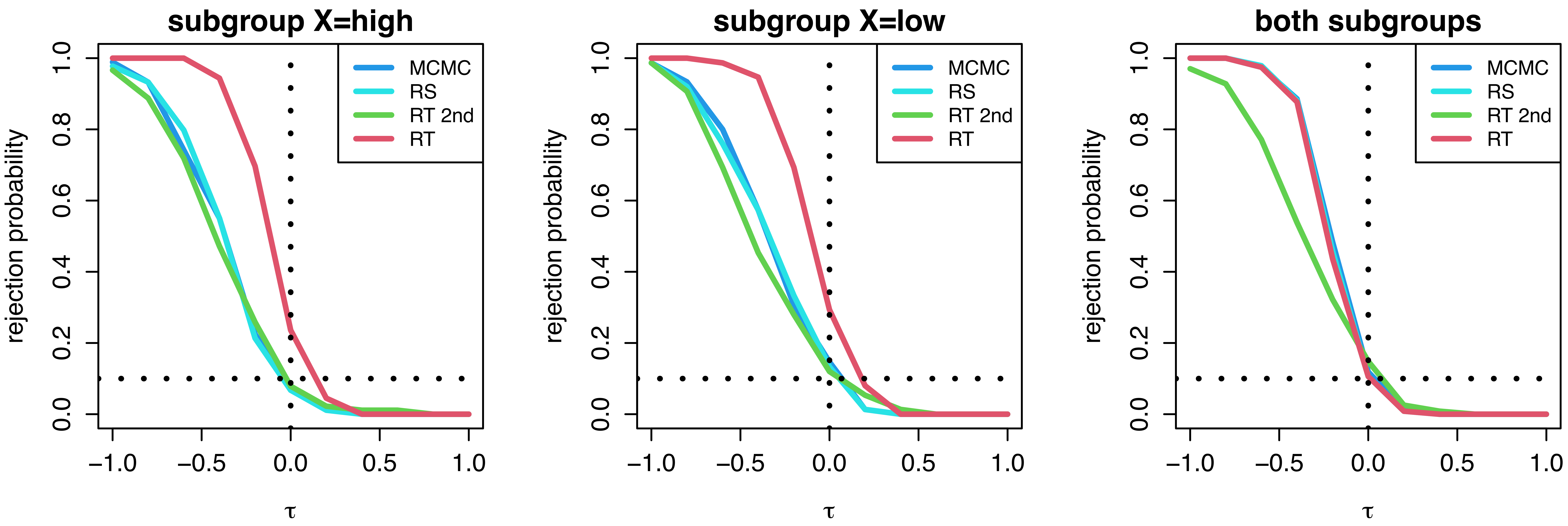

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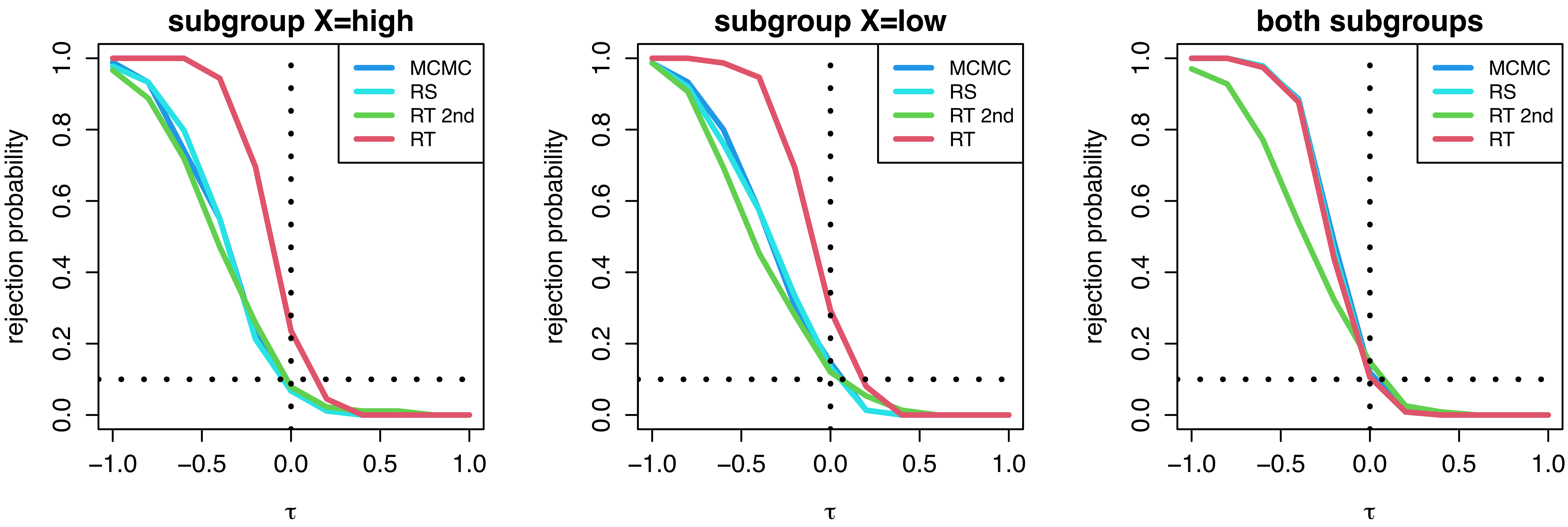

- Type-I error control in every subgroup

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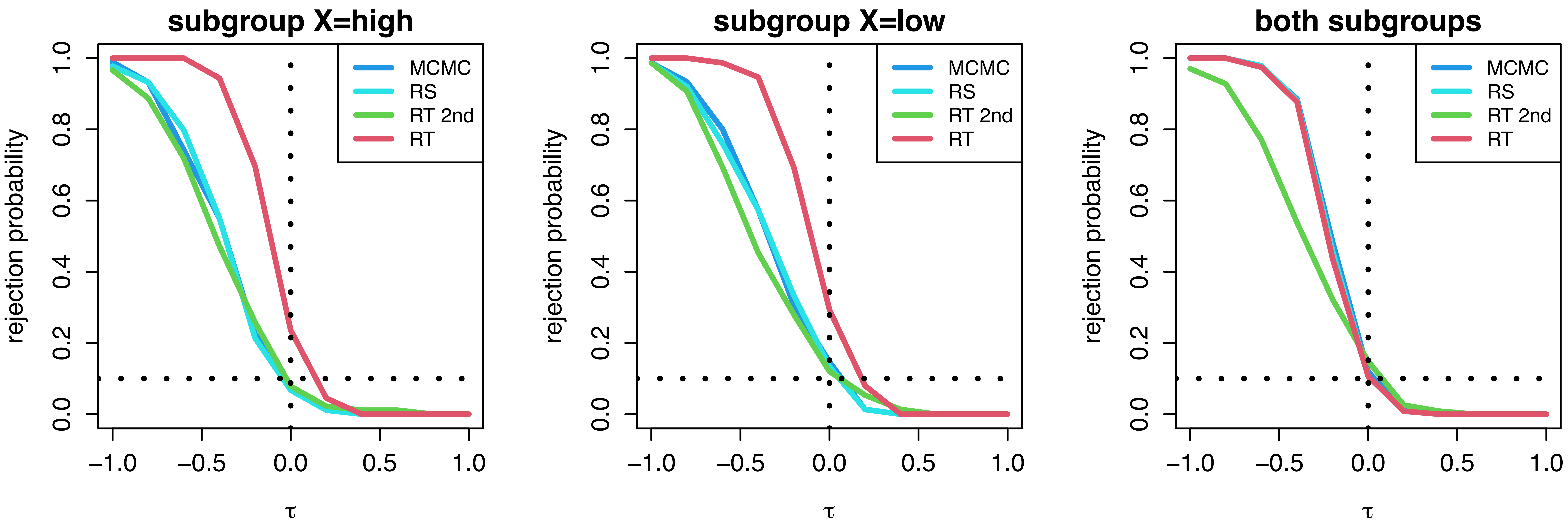

- Type-I error control in every subgroup
- Gain in power when there is a lot of “randomness left”

Conclusion

- Experiments with adaptive treatments, recruitment and null hypothesis
- Visualization via DAGs
- **Key idea: Conditioning randomization p-value on the selection information**
- Computability under general assumptions
- Approximation via rejection sampling or MCMC

Thanks for your attention!

taf40@cam.ac.uk

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Hold-out Units