Optimization-based Sensitivity Analysis

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Overview

1. **Applied Example: NLSYM data**
2. General Framework
3. Sensitivity Analysis for Regression and IV models
4. R-package `optsens`
5. Discussion

Does education make you rich?

Card (1993)

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- National Longitudinal Survey of Young Men (NLSYM)
- From 1966 until 1981
- 3010 individuals

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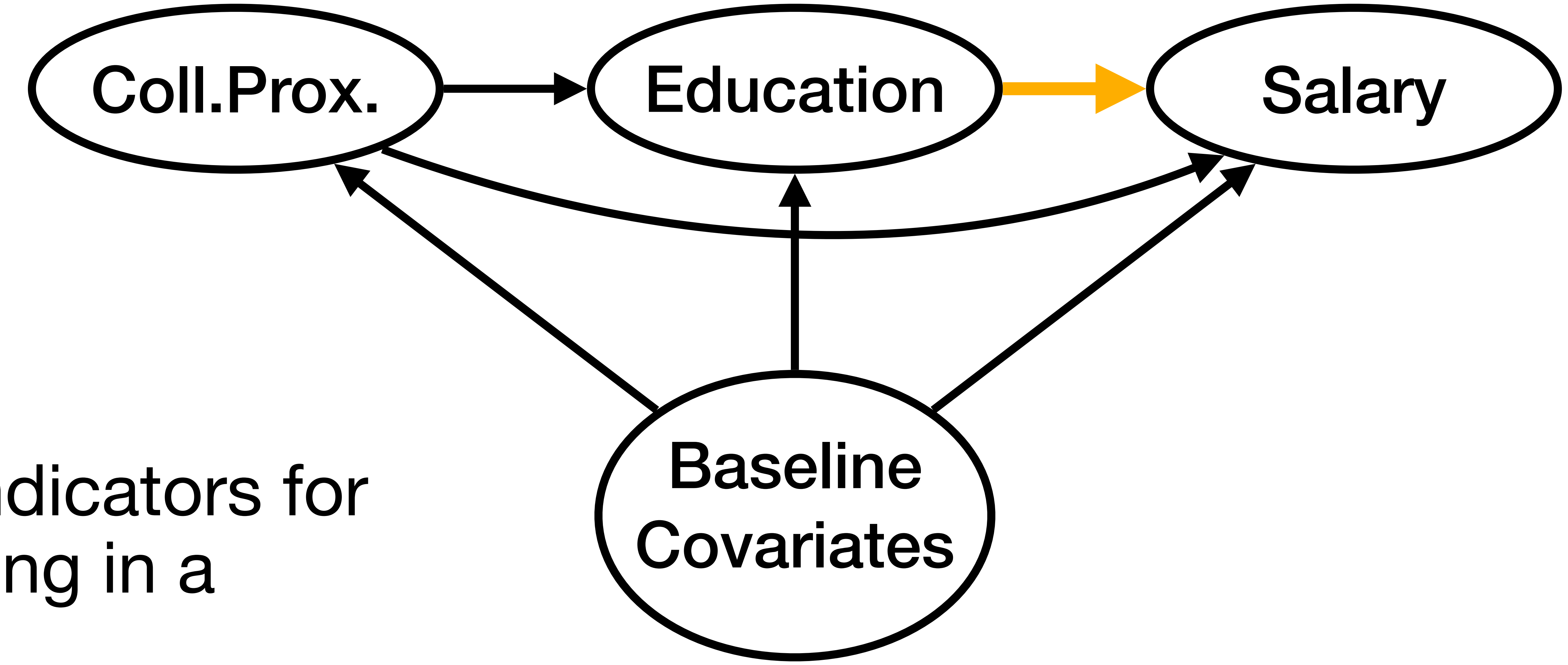
- 3010 individuals
- Y: log-salary
- D: years of schooling
- X: years of labour force experience; indicators for living in the south, being black and living in a metropolitan area
- Z: college proximity



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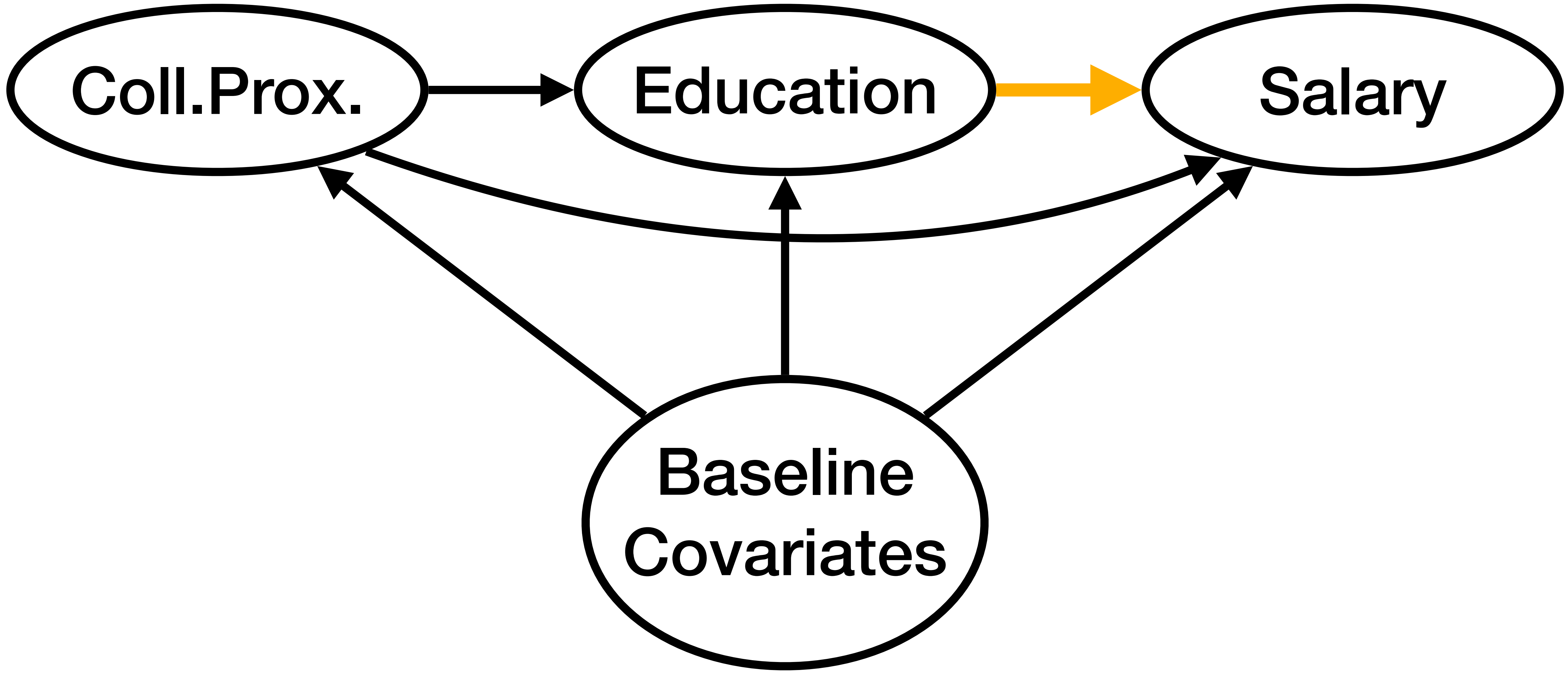

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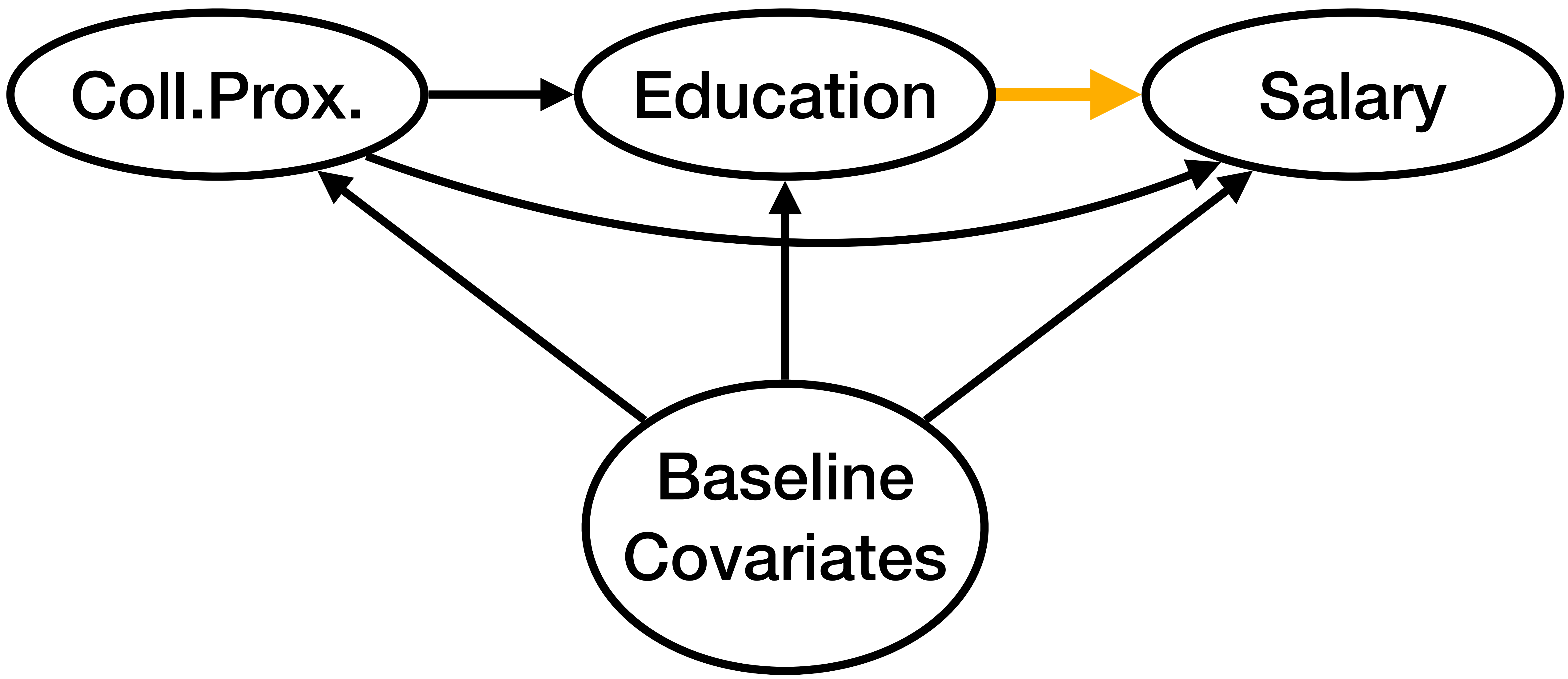
- Causal question $\rightarrow$ Linear causal effect β



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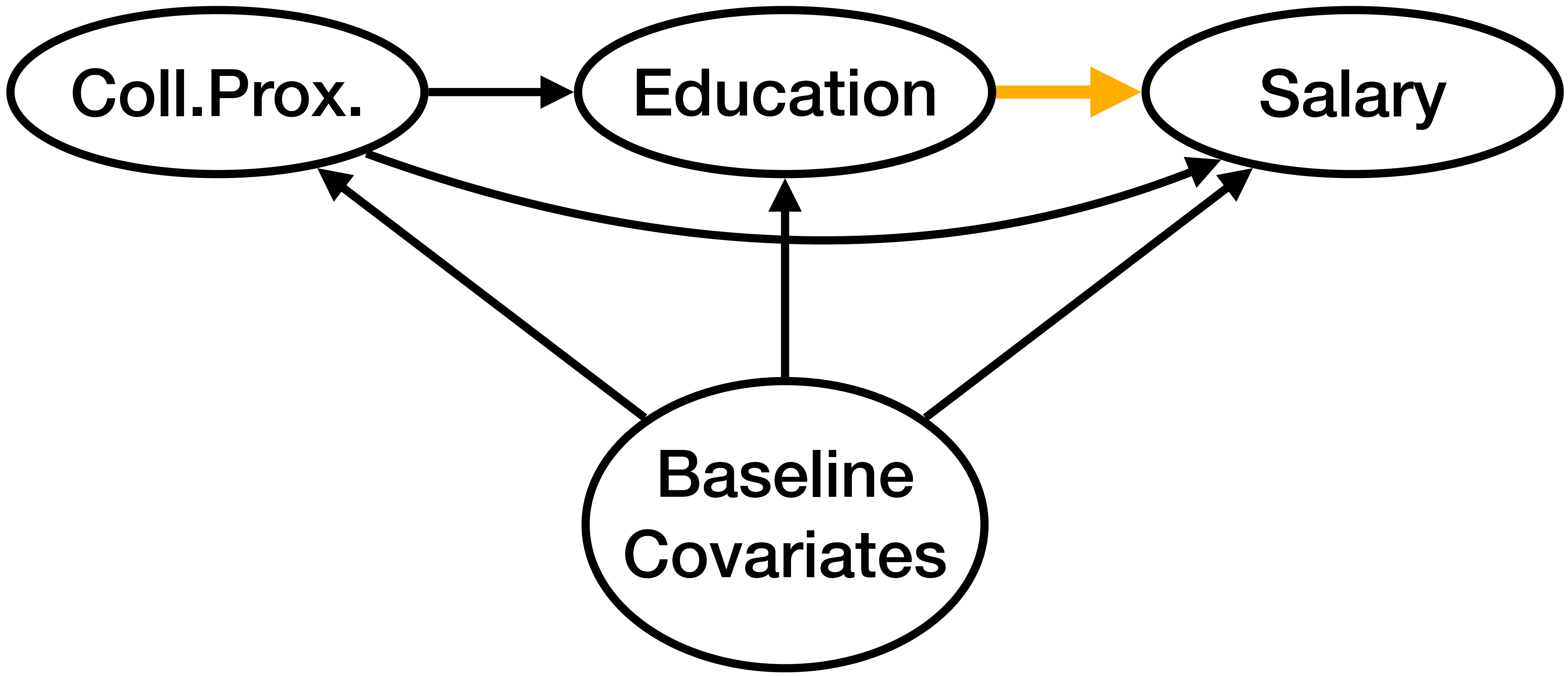

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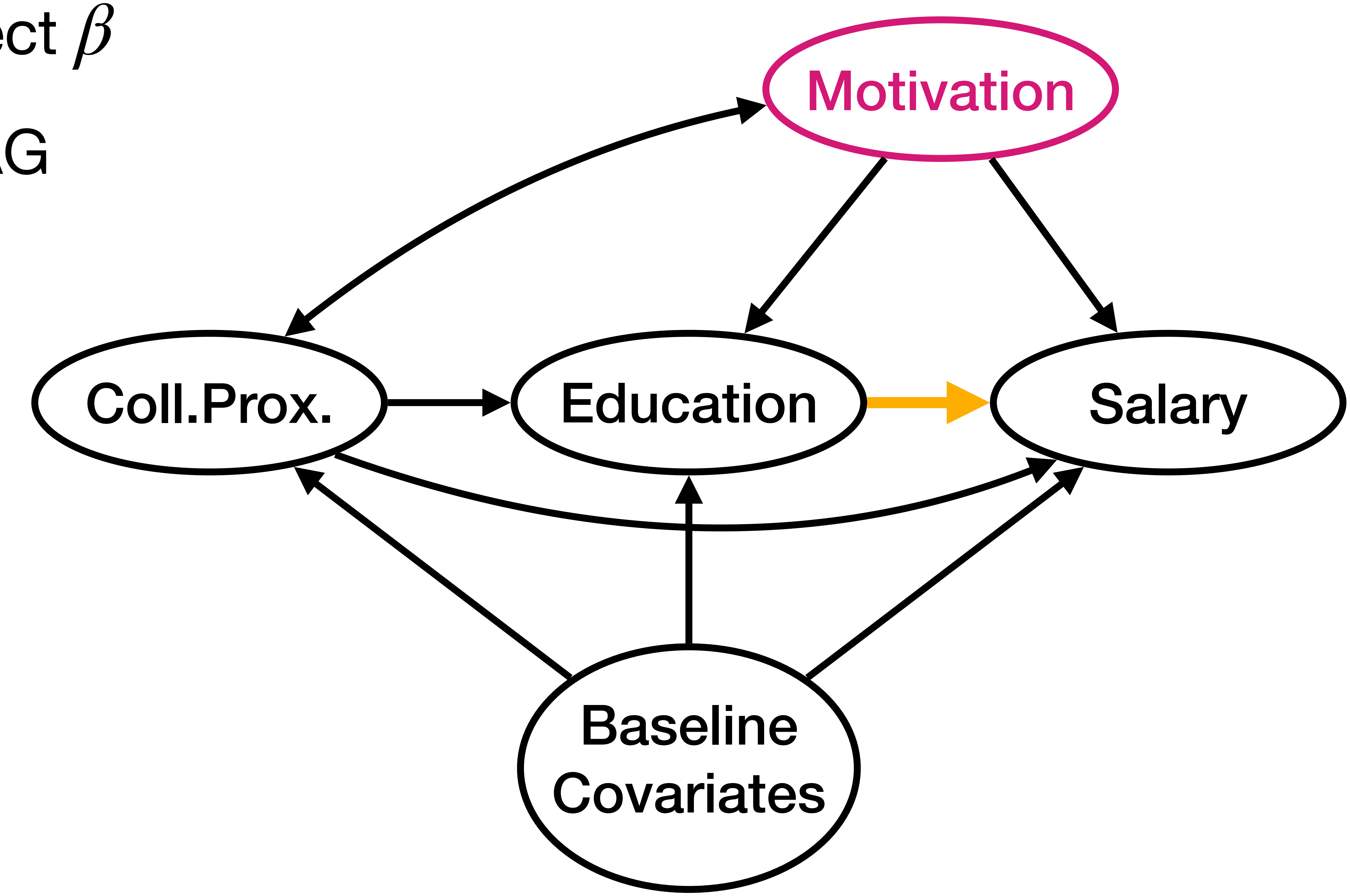

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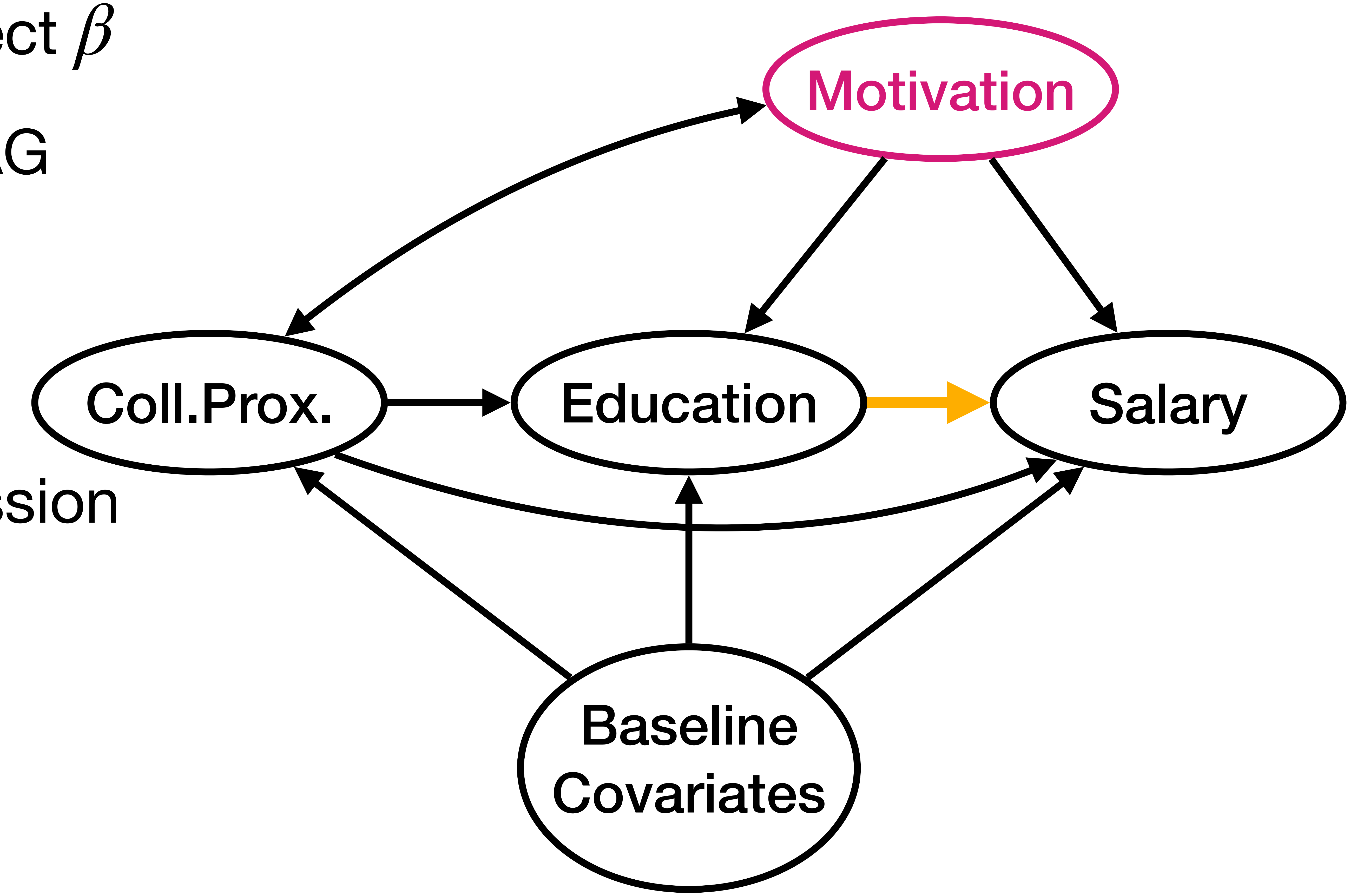
- Causal question $\rightarrow$ Linear causal effect β
- Data-generating process follows the DAG
- Obstacle: unmeasured variables U
- If U was observed, fit regression model $Y \sim D + X + Z + U$ and get the regression coefficient of D:

$$\beta = \beta_{Y \sim D | X, Z, U}$$



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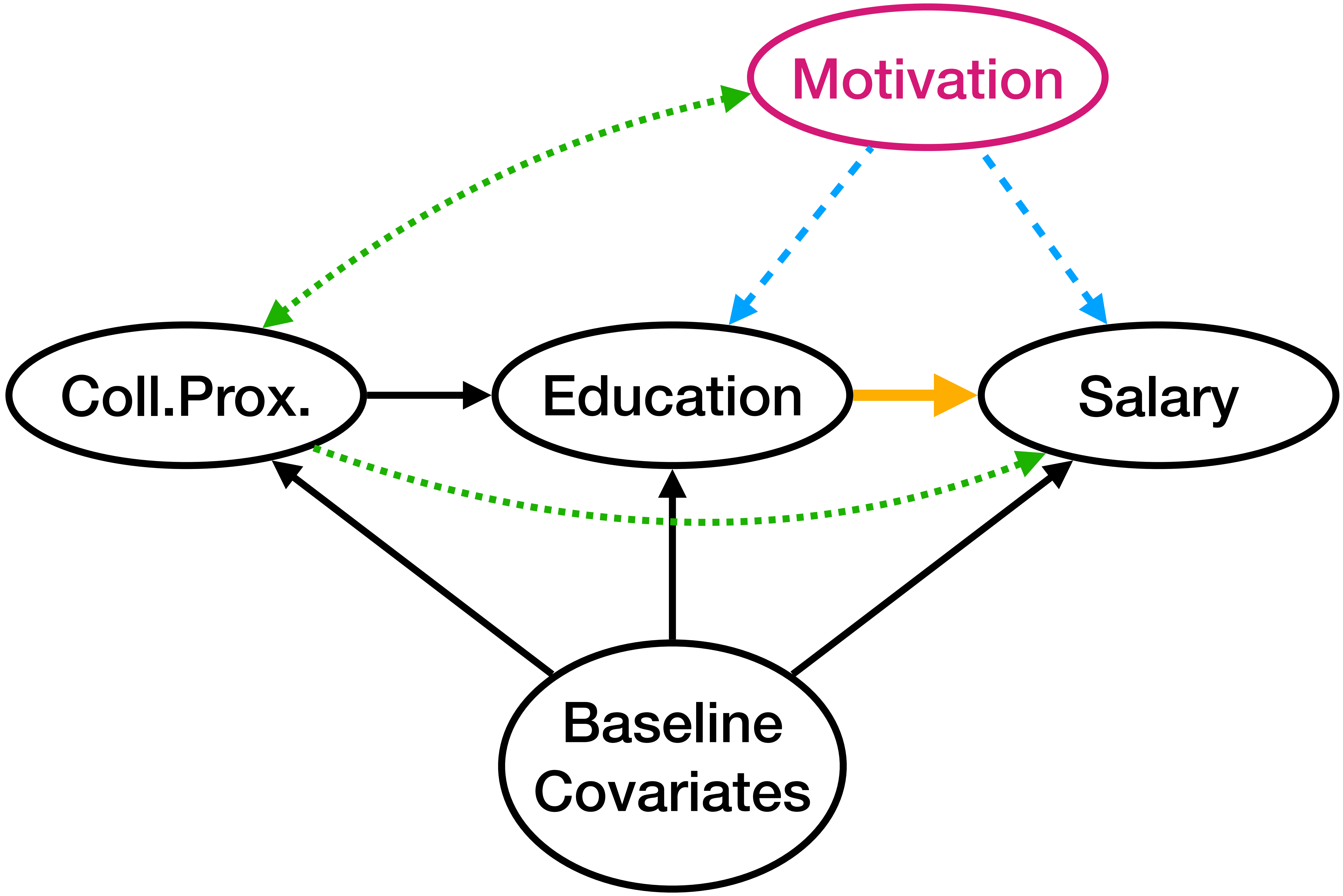

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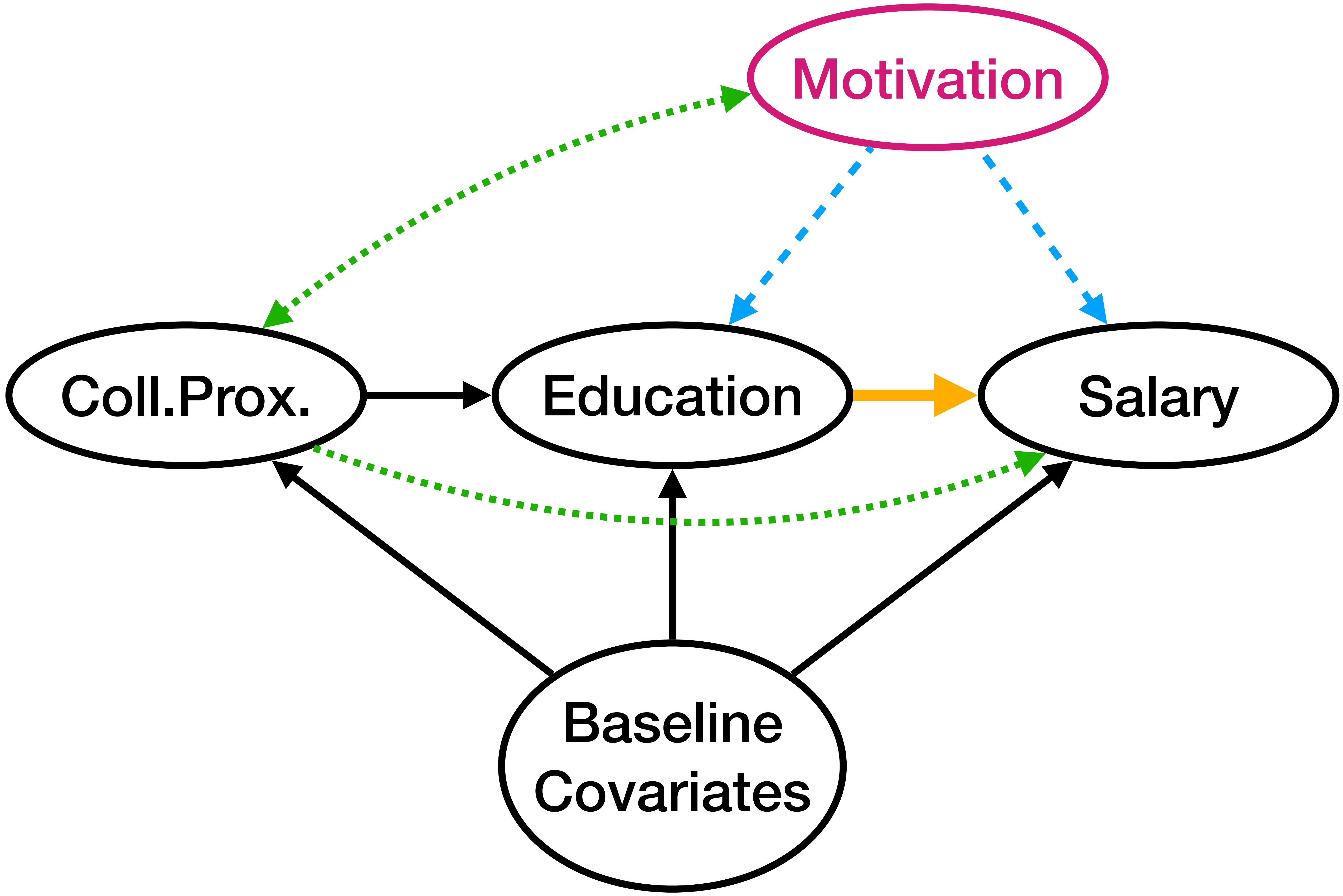

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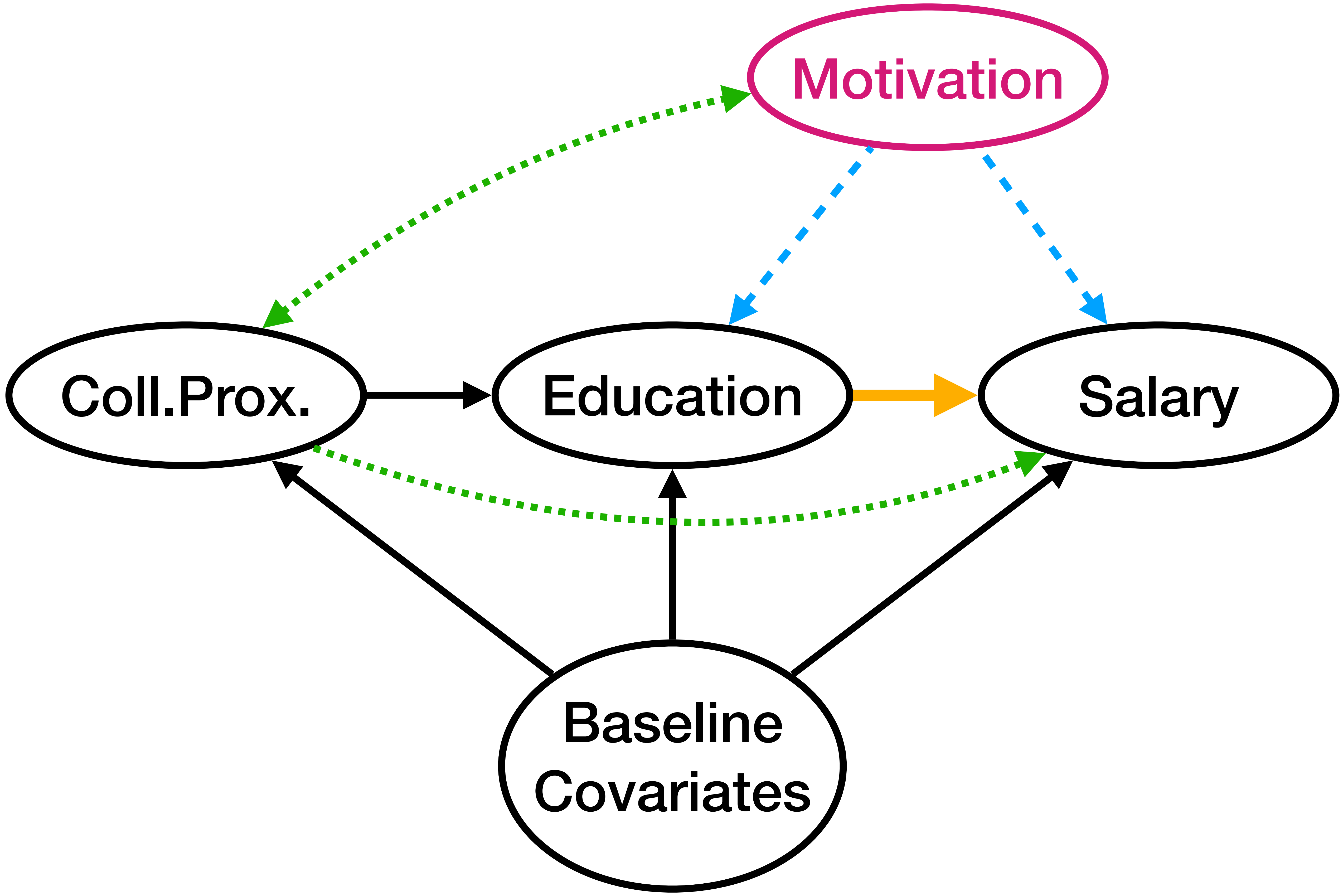

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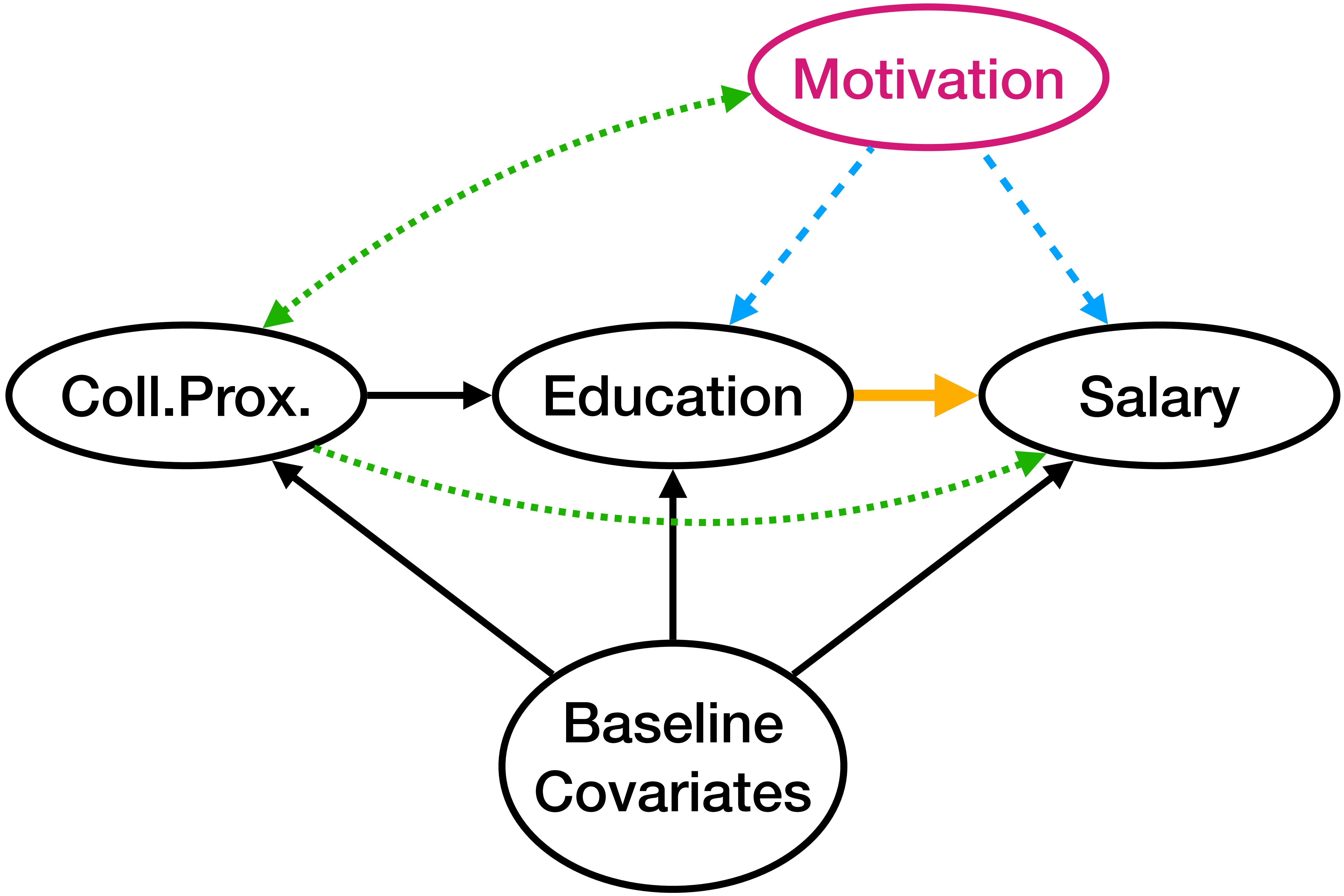

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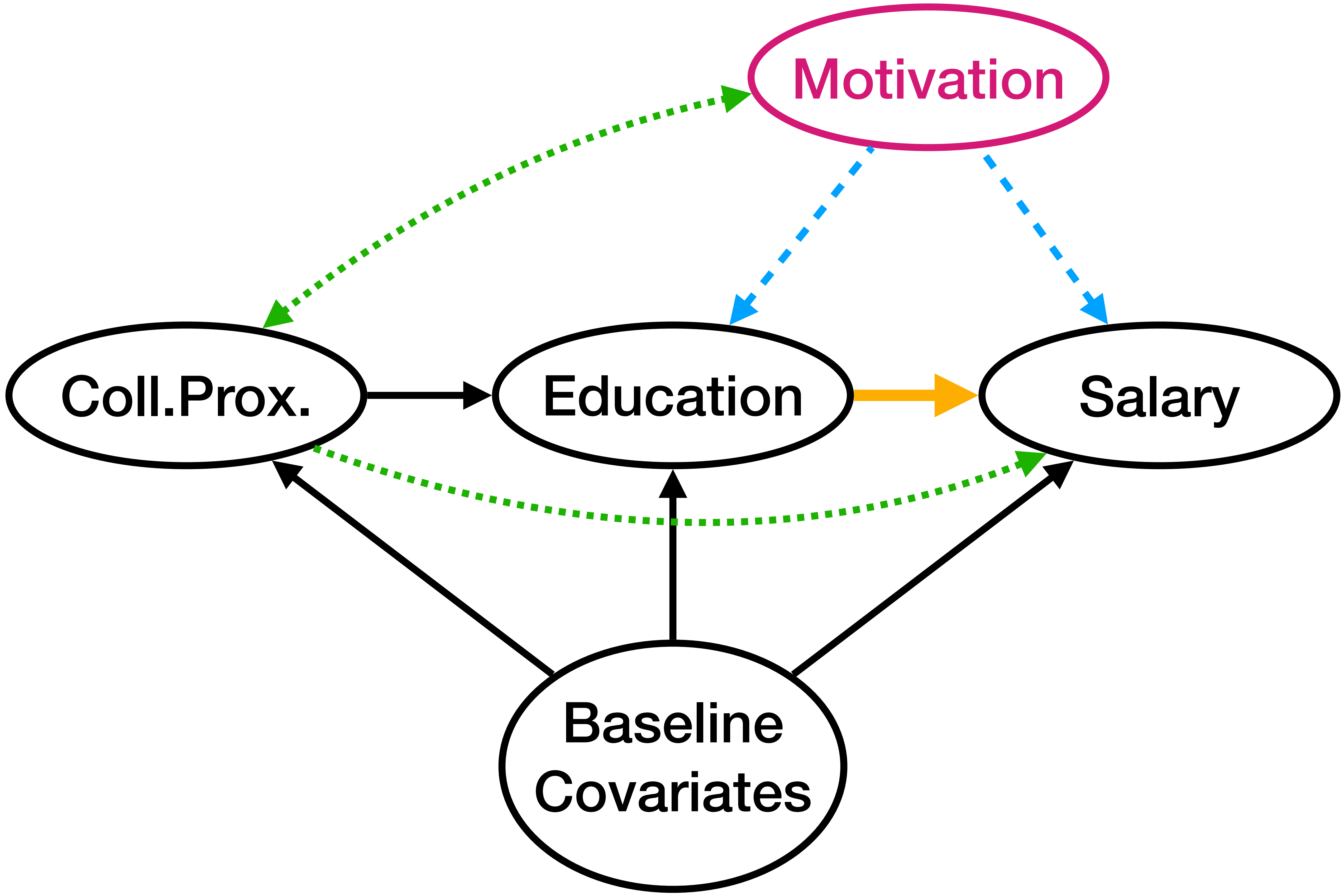

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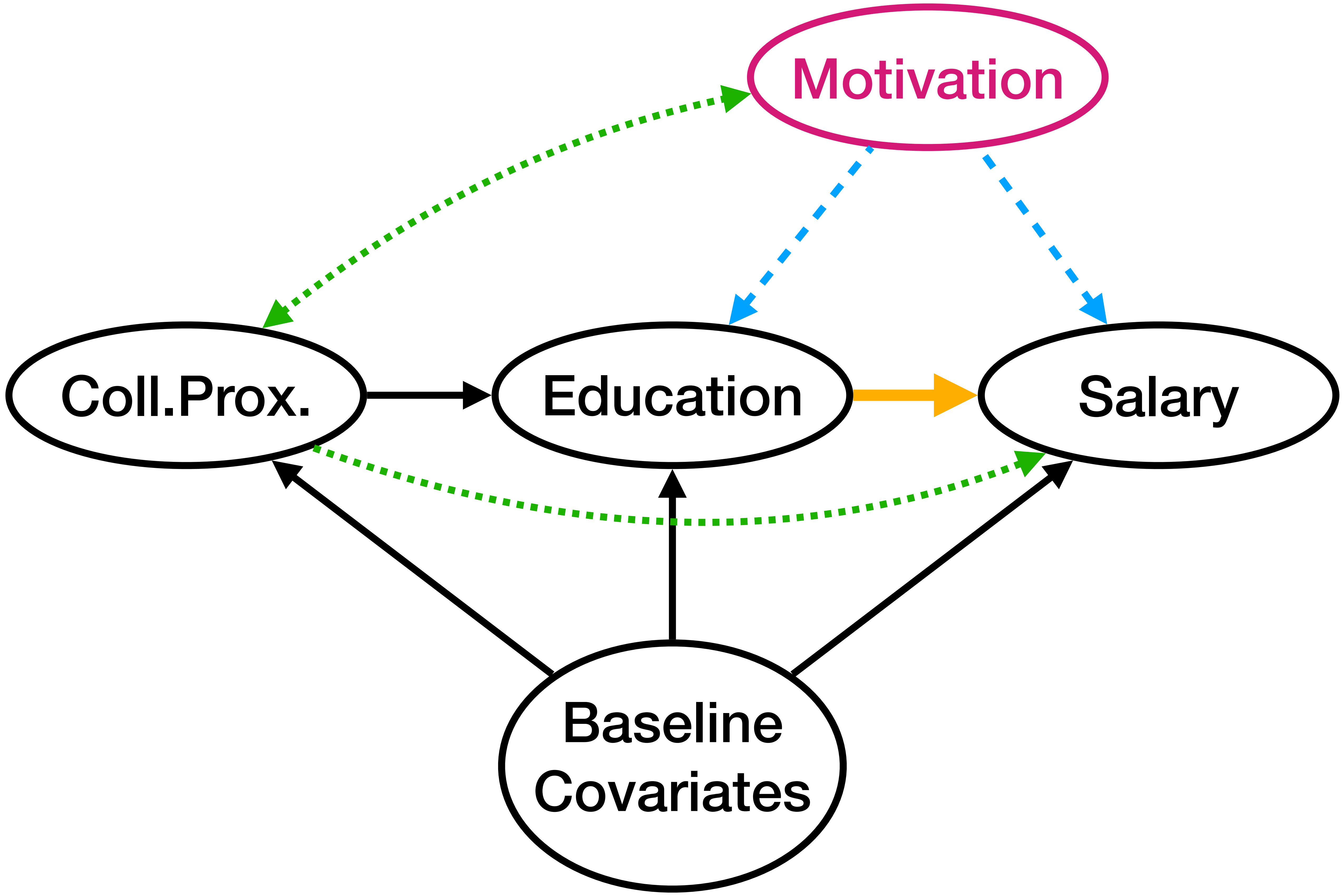

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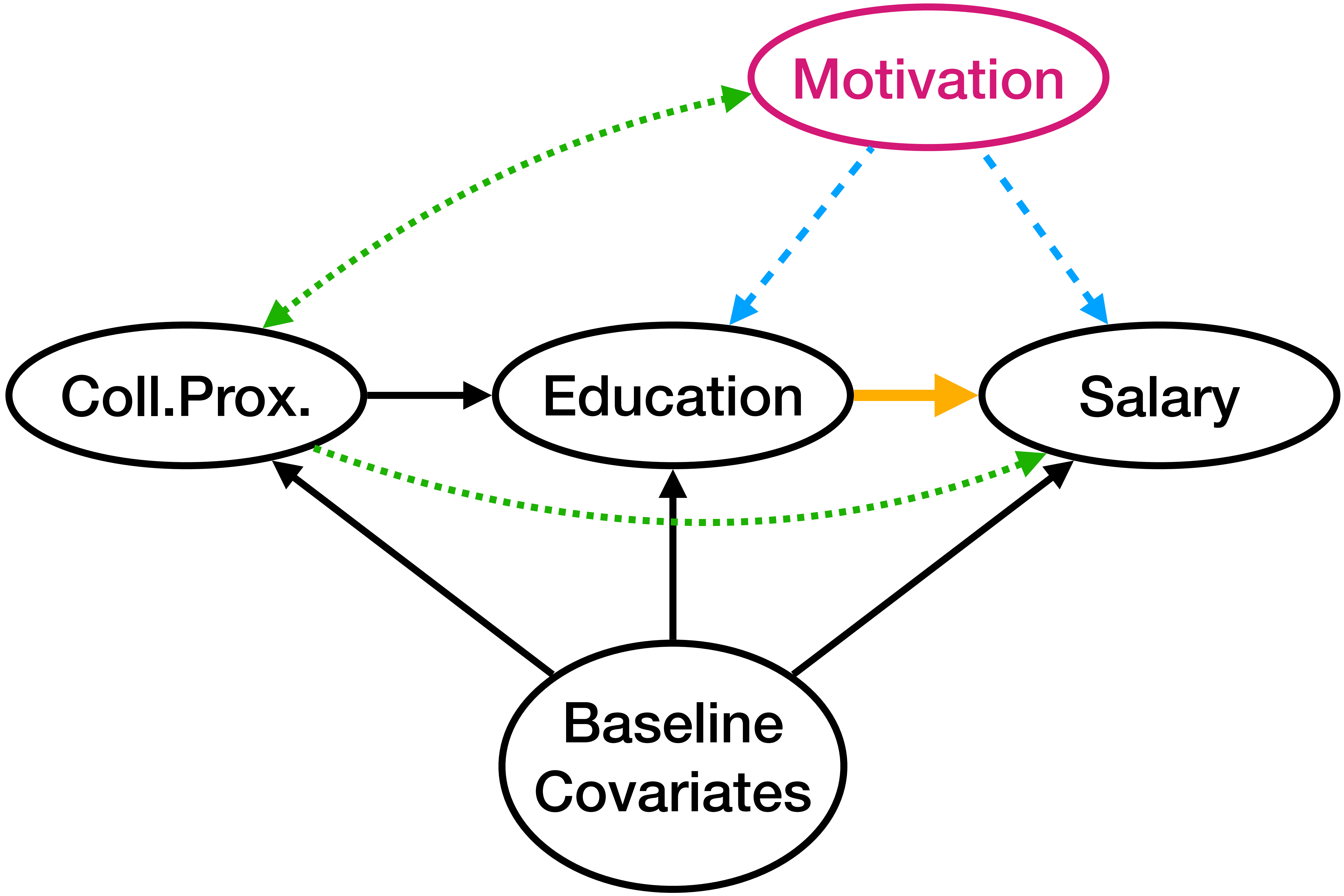

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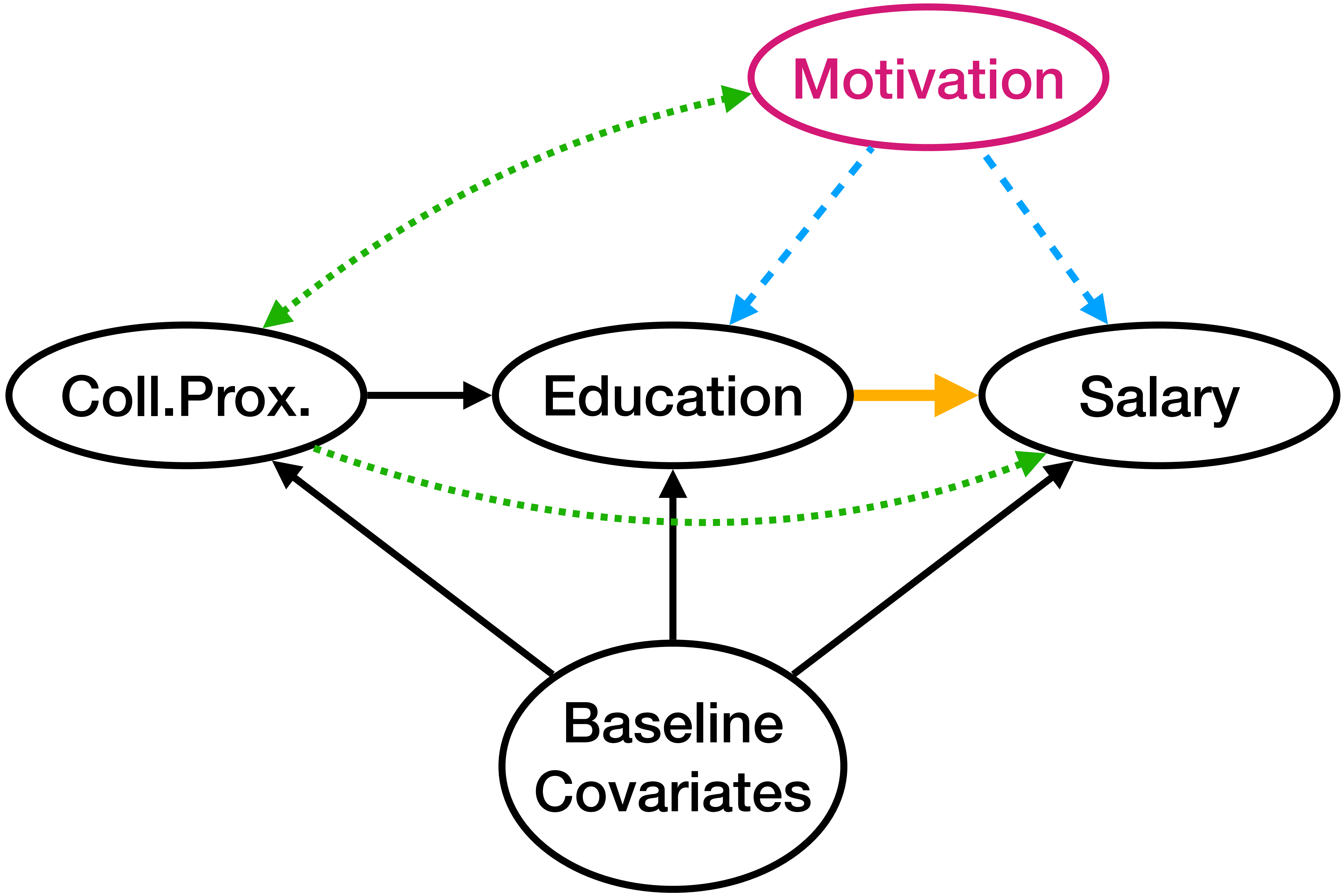

We should do sensitivity analysis!

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Sensitivity Analysis as an Optimization Problem

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  - Rosenbaum (1987)
  - Balke & Pearl (1994, 1997)
  - Ding & VanderWeele (2016)
  - Etc.

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- **Our contribution:** Take the optimization perspective seriously

  → solve numerically instead of analytically

Estimating the PIR & Sensitivity Intervals

Imbens & Manski (2004)

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- $(1-\alpha)$ Sensitivity Interval: $(1-\alpha)$ confidence interval for the PIR
- Bootstrap approach:
  - Create bootstrap samples → Compute bootstrap estimators $\hat{\hat{\theta}}$
  - Solve the optimization problems $\nu(\hat{\hat{\theta}})$
  - Compute the quantiles via percentile, basic or BCa bootstrap

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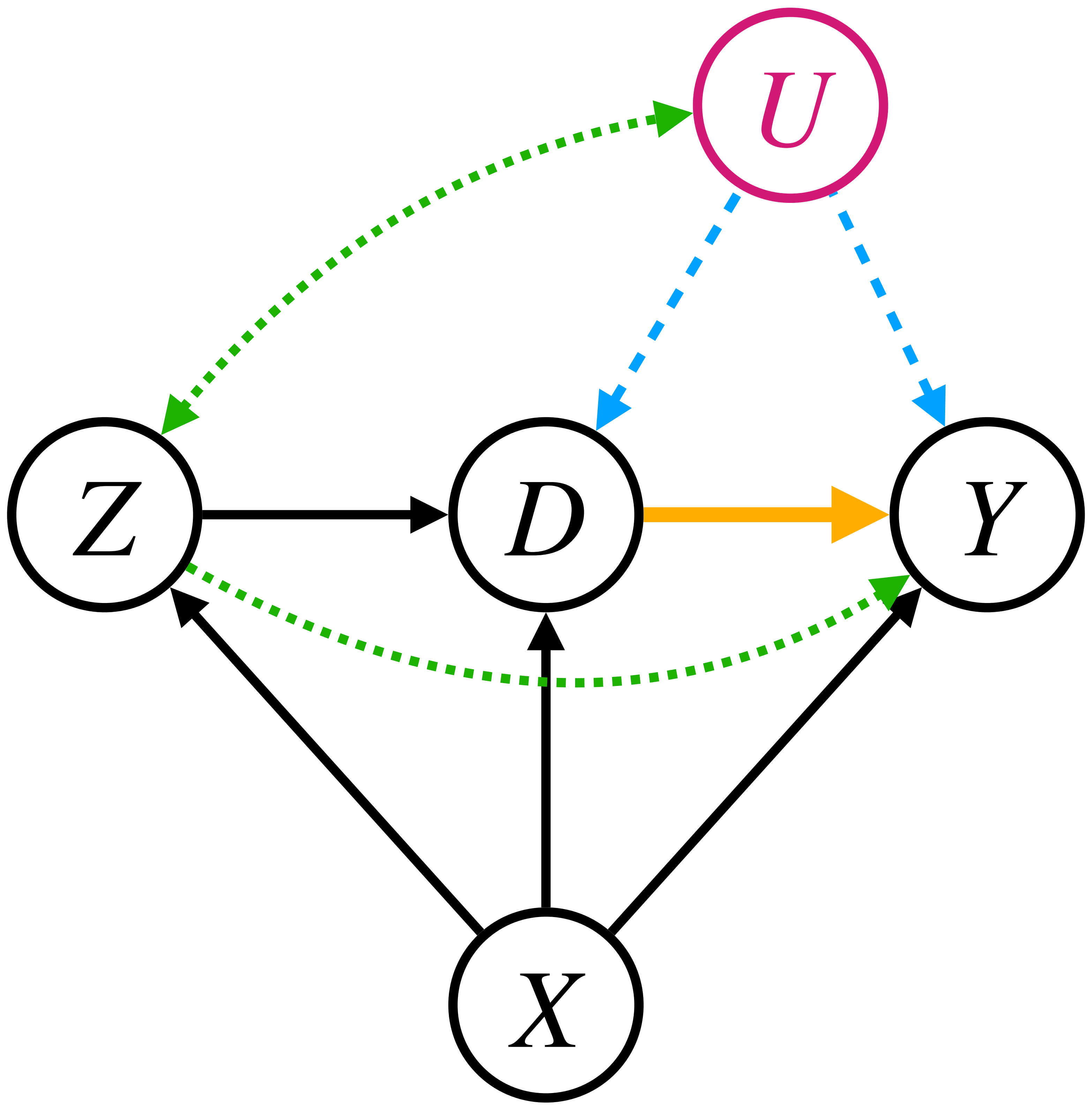
- Algebraic relationships between different R- and R^2-values: **R^2-calculus**

Identification of the Causal Effect



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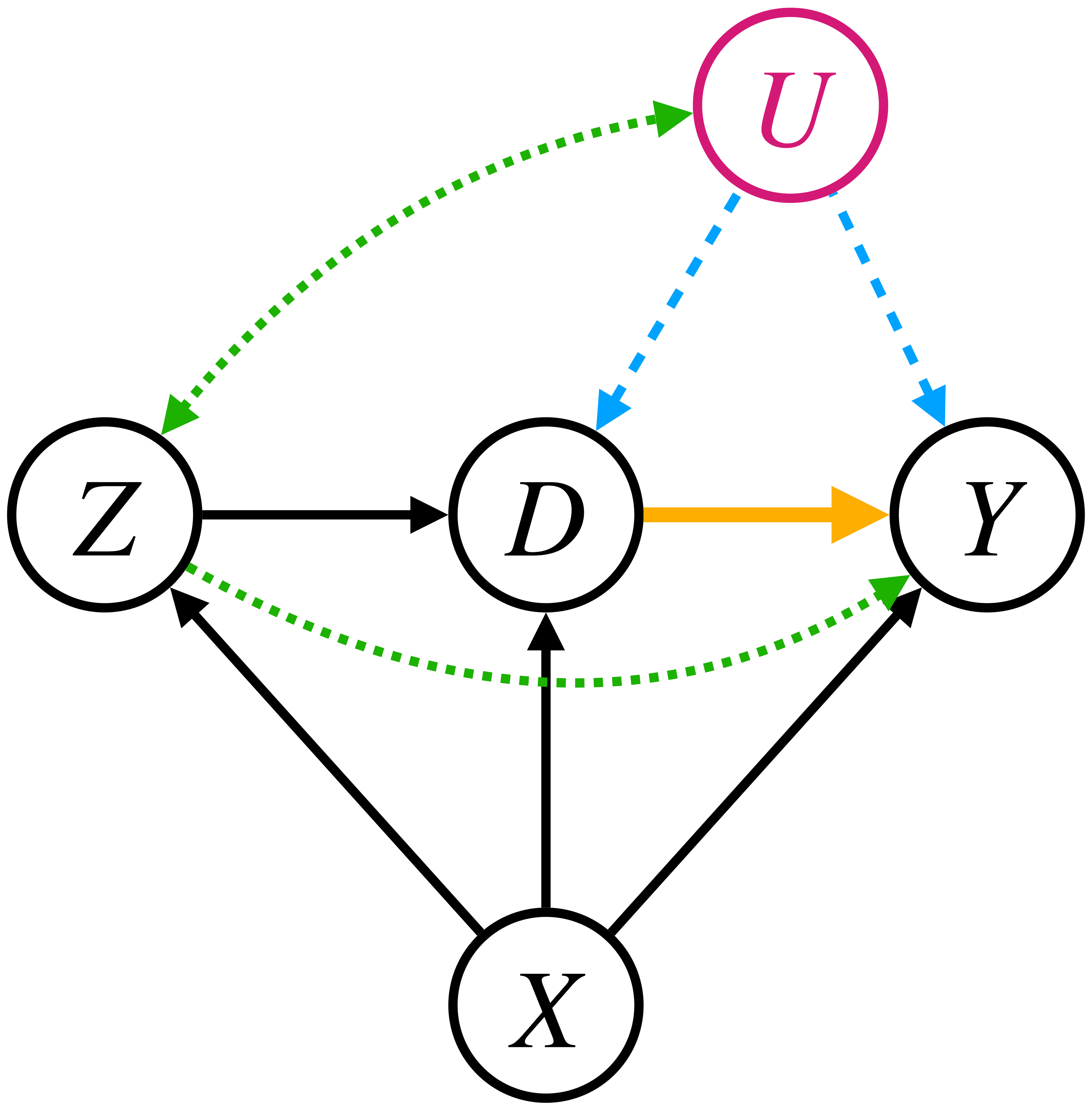

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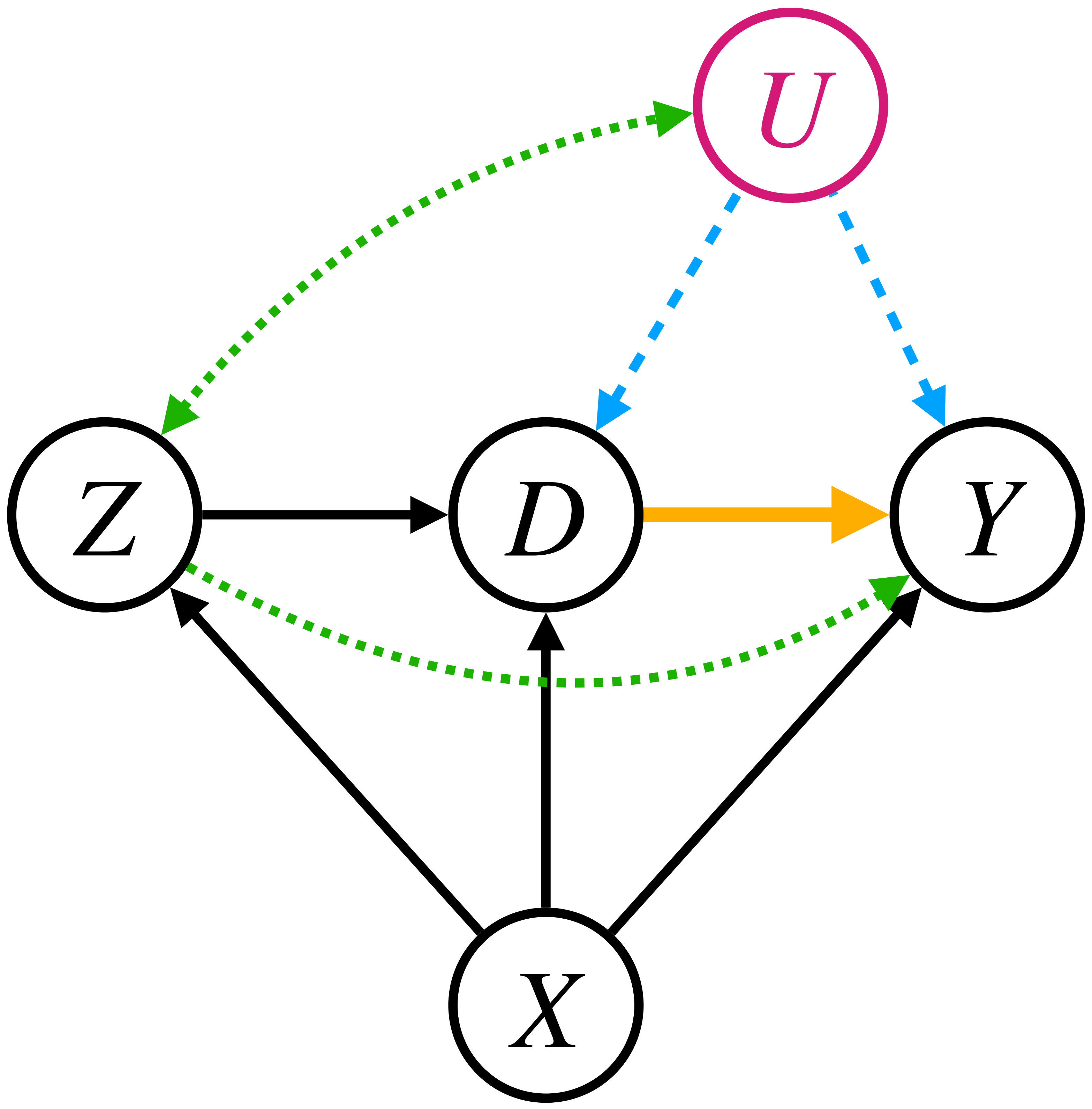

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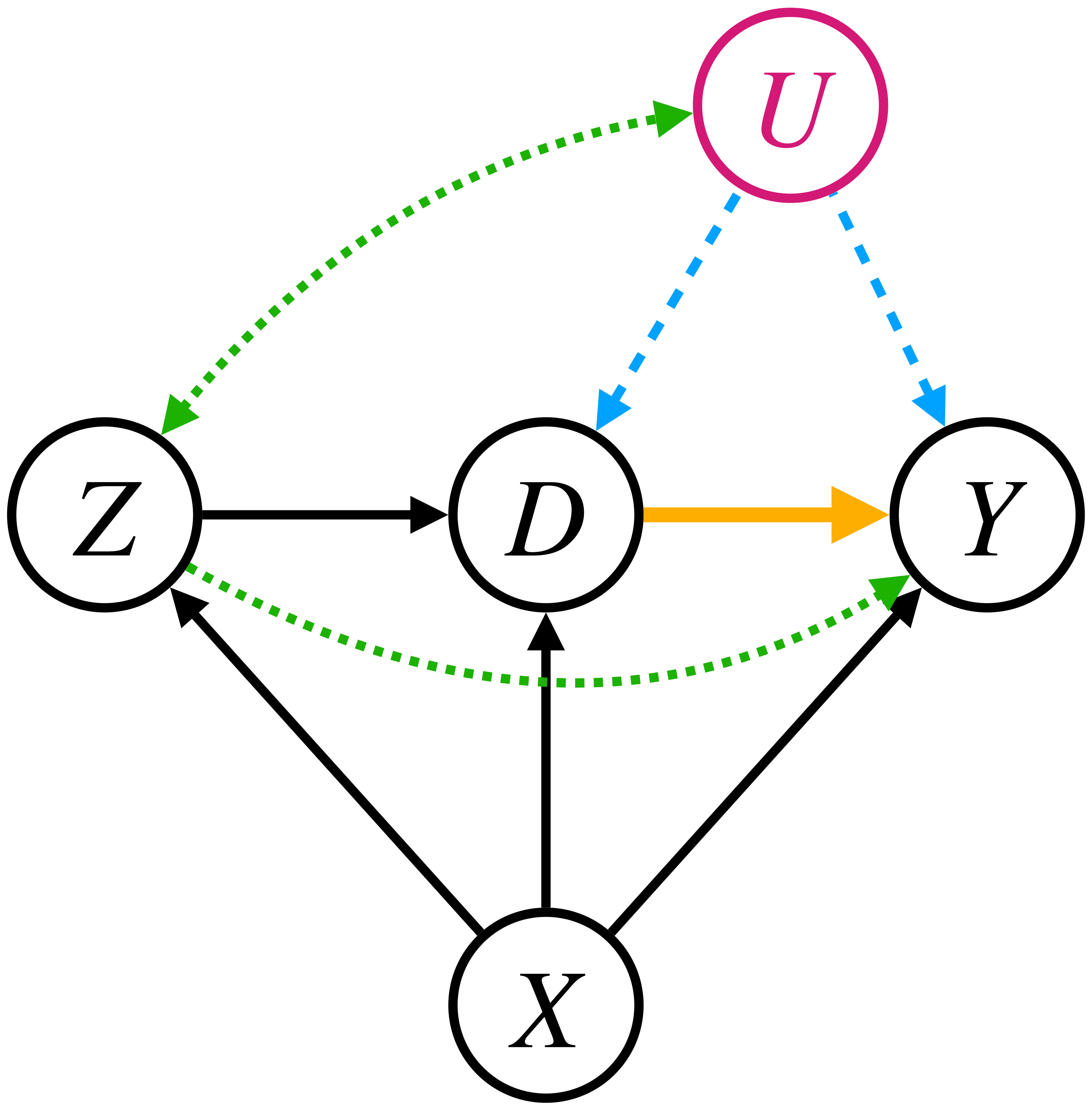

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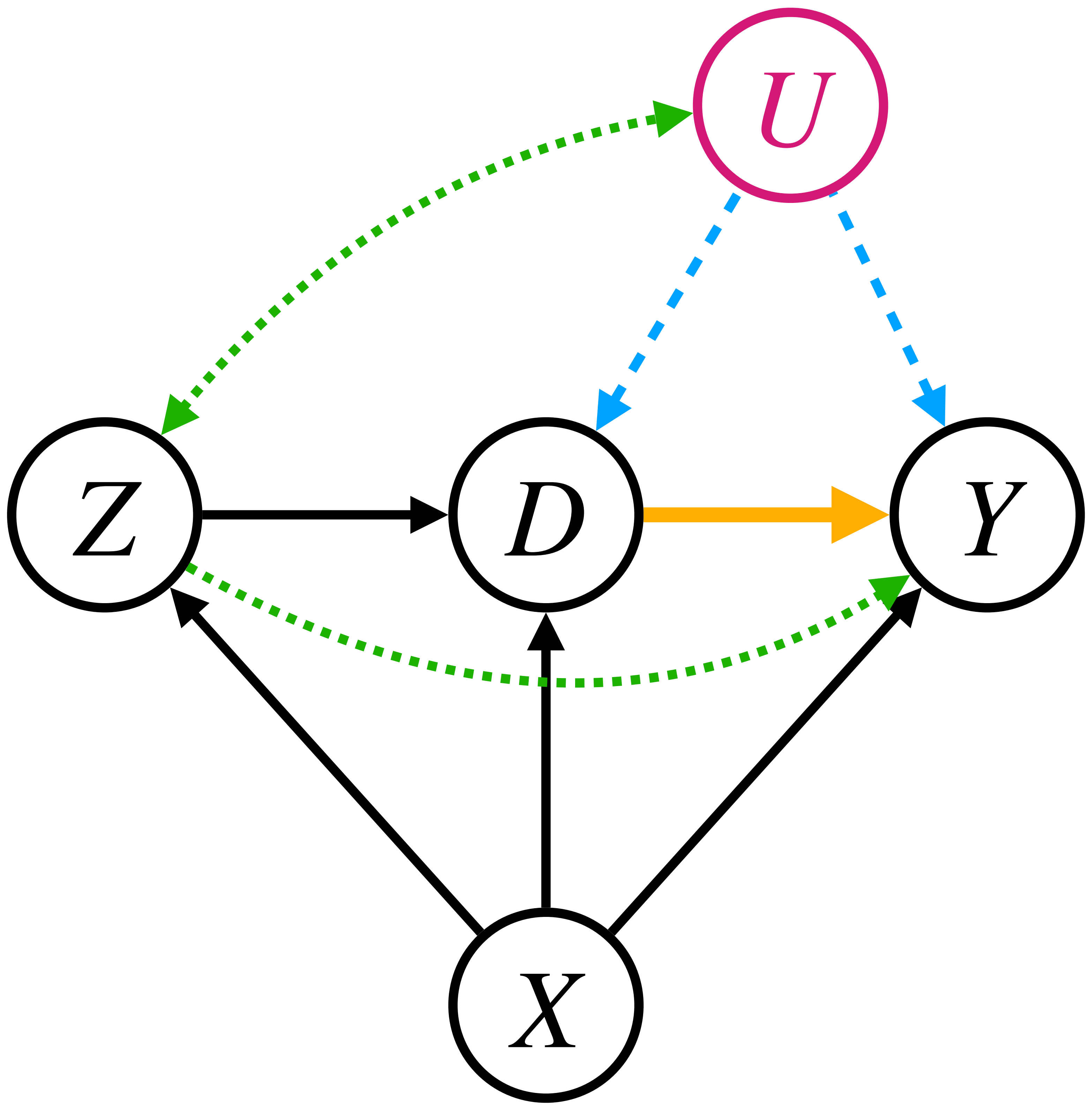

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- Estimable parameters θ: $\beta_{Y \sim D|X,Z}$, $\mathrm{sd}(Y^{\perp X,Z,D})$, $\mathrm{sd}(D^{\perp X,Z})$, etc.
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$$f_{Y \sim Z|X,U,D}\sqrt{1 - R^2_{Y \sim U|X,Z,D}} = f_{Y \sim Z|X,D}\sqrt{1 - R^2_{Z \sim U|X,D}} - R_{Y \sim U|X,Z,D}\, R_{Z \sim U|X,D}$$

$$f_{Z \sim U|X,D}\sqrt{1 - R^2_{D \sim U|X,Z}} = f_{Z \sim U|X}\sqrt{1 - R^2_{D \sim Z|X}} - R_{D \sim Z|X}\, R_{D \sim U|X,Z} \qquad (\star)$$

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Direct Bounds

- Directly specify range of values, e.g.

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- Assumption: $X = (\dot{X}, \tilde{X})$ such that $R^2_{U\sim\dot{X}|\tilde{X},Z} = 0$

$$R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z} \leq 2\, R^2_{Y\sim\dot{X}_j|\tilde{X},\dot{X}_{-j},Z}$$

- Meaning: *The unmeasured confounder U can explain at most twice as much variation in Y as $\dot{X}_j$ does — after regressing out the linear effects of $(\tilde{X}, \dot{X}_{-j}, Z)$.*

Sensitivity Model $\Psi(\theta)$

$U \rightarrow D$	$R_{D\sim U\|X,Z} \in [B^l_{UD}, B^u_{UD}]$
	$R^2_{D\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UD}\, R^2_{D\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
$U \rightarrow Y$	$R_{Y\sim U\|X,D,Z} \in [B^l_{UY}, B^u_{UY}]$
	$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
	$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z,D} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	$R_{Y\sim Z\|X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Z\sim U\|\tilde{X},\dot{X}_{-j}} \leq b_{UZ}\, R^2_{Z\sim \dot{X}_j\|\tilde{X},\dot{X}_{-j}}$
$Z \rightarrow Y$	$R_{Y\sim Z\|X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
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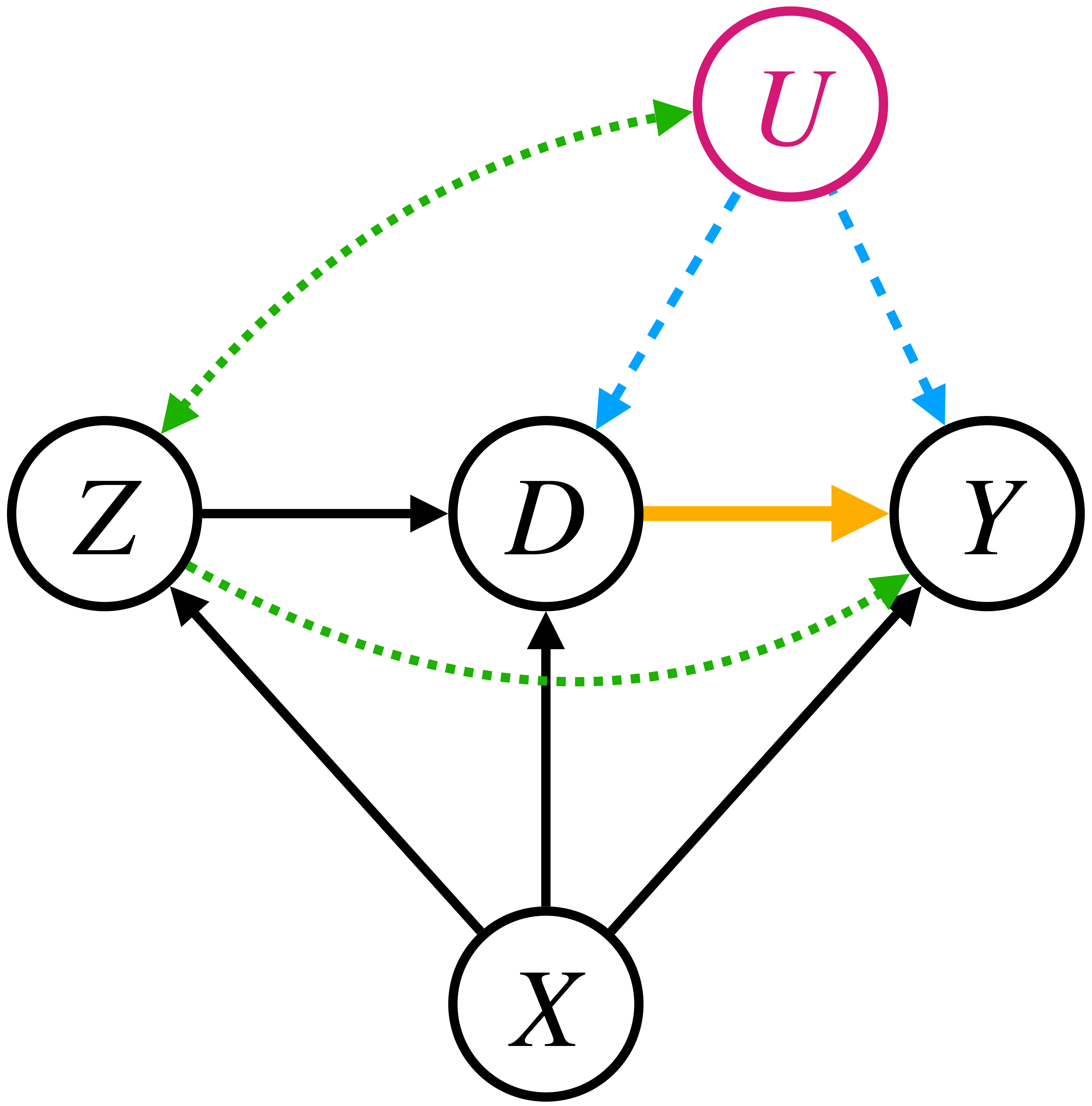

Cinelli & Hazlett (2020)

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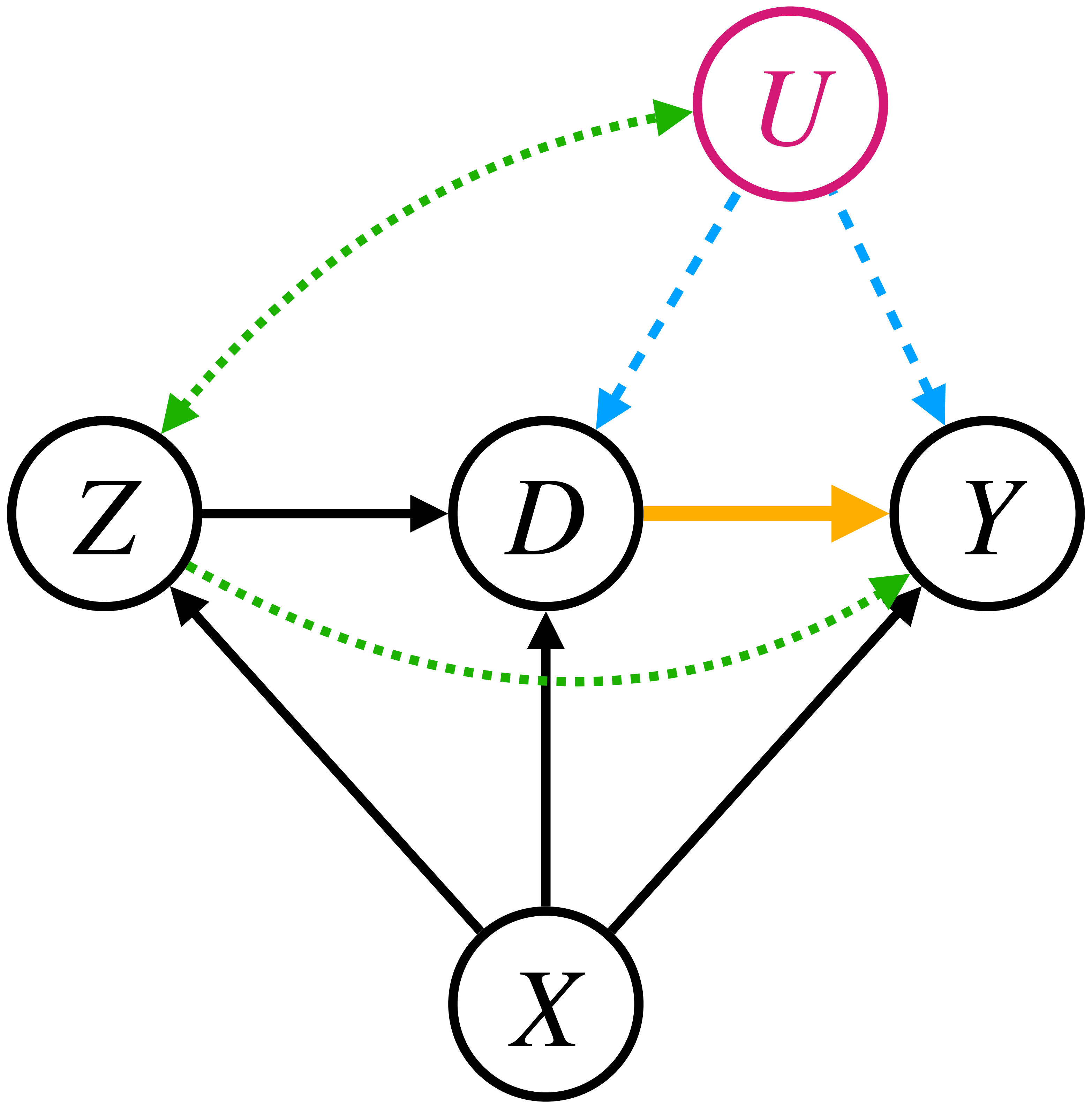
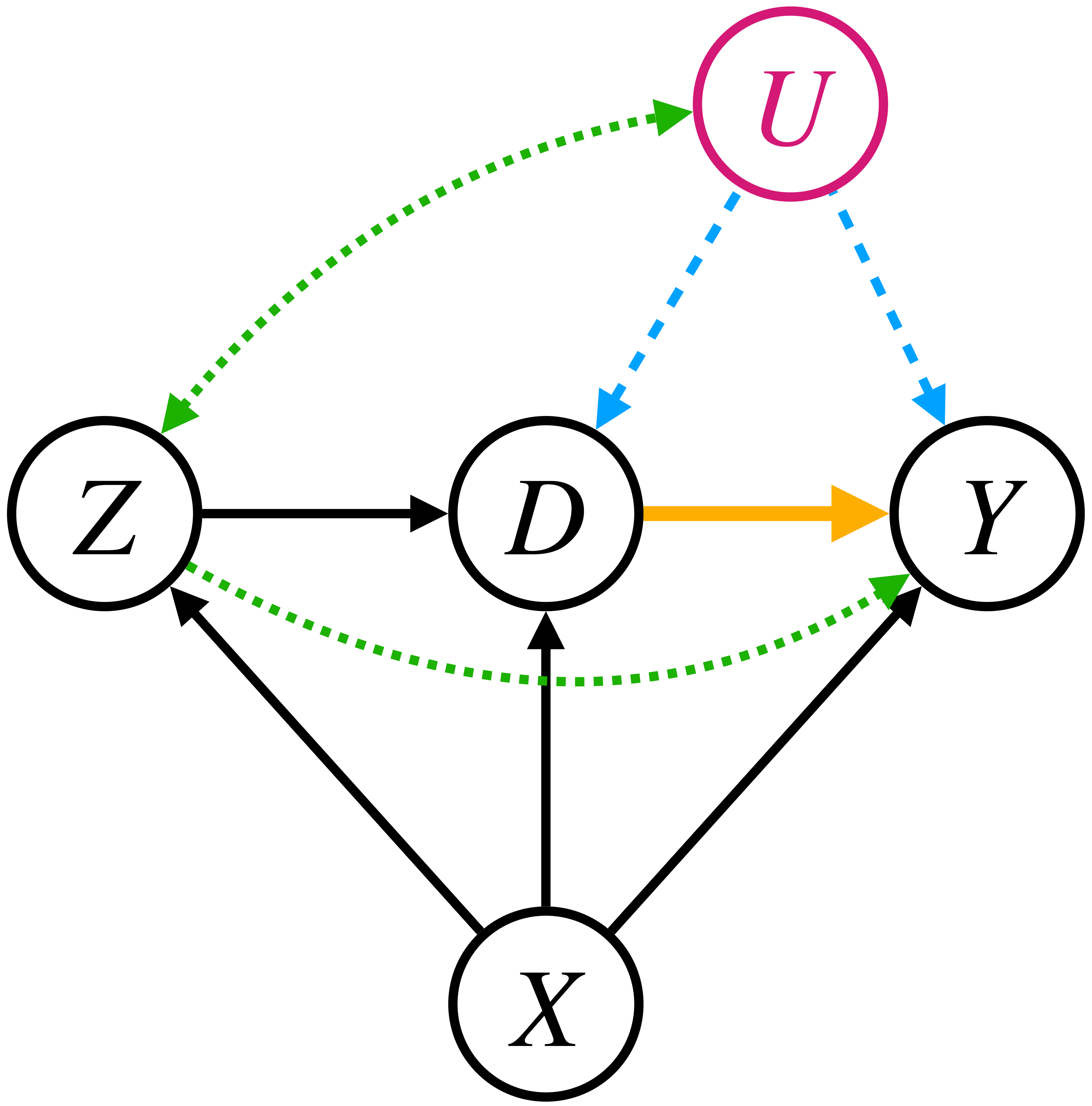


Cinelli & Hazlett (2020)

Sensitivity Model $\Psi(\theta)$

$U \to D$	$R_{D\sim U\vert X,Z} \in [B^l_{UD}, B^u_{UD}]$
	$R^2_{D\sim U\vert\tilde{X},\dot{X}_I,Z} \leq b_{UD}\, R^2_{D\sim\dot{X}_J\vert\tilde{X},\dot{X}_I,Z}$
$U \to Y$	$R_{Y\sim U\vert X,D,Z} \in [B^l_{UY}, B^u_{UY}]$
	$R^2_{Y\sim U\vert\tilde{X},\dot{X}_I,Z} \leq b_{UY}\, R^2_{Y\sim\dot{X}_J\vert\tilde{X},\dot{X}_I,Z}$
	$R^2_{Y\sim U\vert\tilde{X},\dot{X}_I,Z,D} \leq b_{UY}\, R^2_{Y\sim\dot{X}_J\vert\tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	$R_{Y\sim Z\vert X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Z\sim U\vert\tilde{X},\dot{X}_{-j}} \leq b_{UZ}\, R^2_{Z\sim\dot{X}_j\vert\tilde{X},\dot{X}_{-j}}$
$Z \to Y$	$R_{Y\sim Z\vert X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Y\sim Z\vert X,U,D} \leq b_{ZY}\, R^2_{Y\sim\dot{X}_j\vert\tilde{X},\dot{X}_{-j},Z,U,D}$

$+(\star)$



Cinelli & Hazlett (2020)

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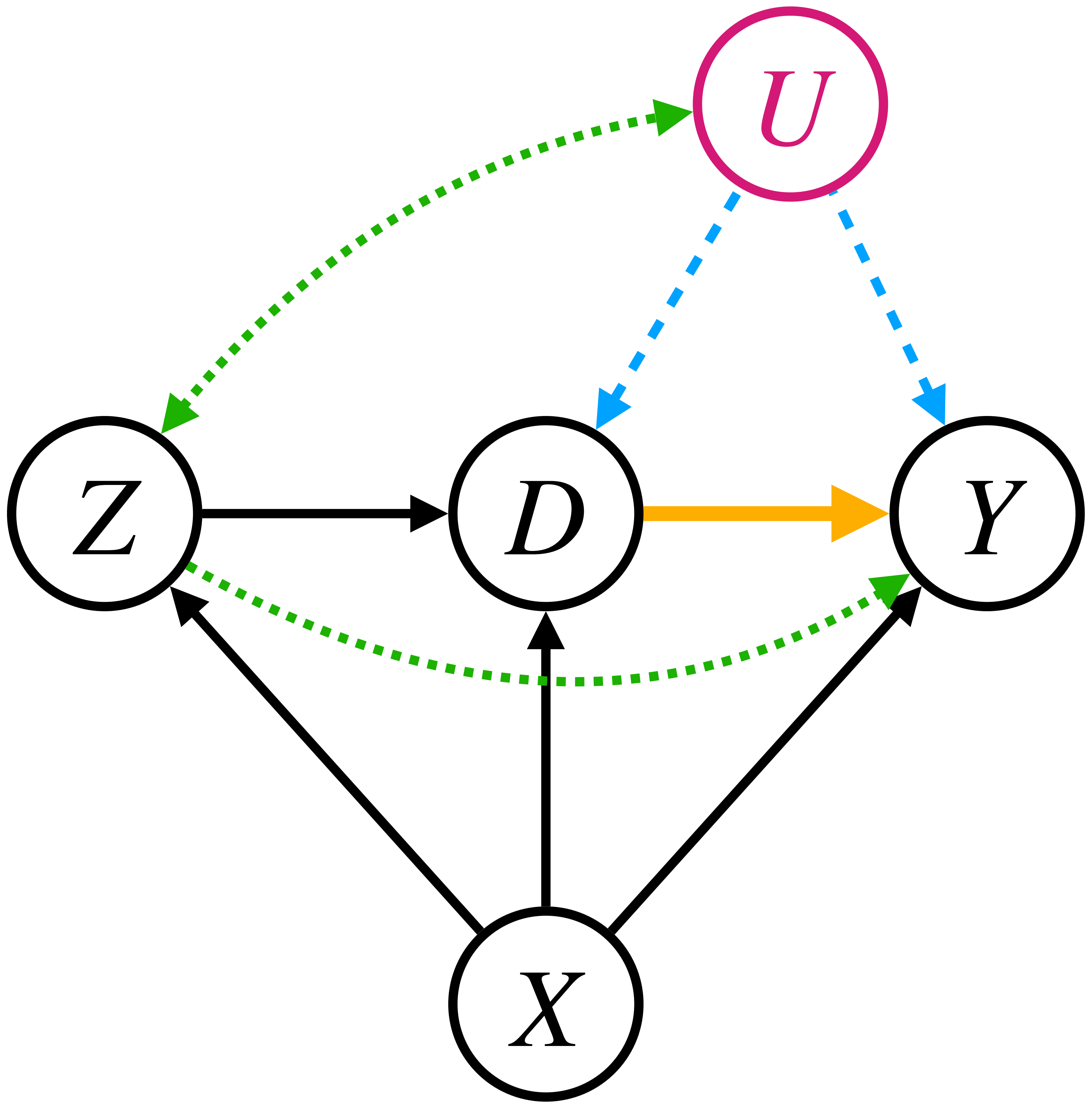
$U \to Y$	$R_{Y\sim U\mid X,D,Z} \in [B^l_{UY}, B^u_{UY}]$
	$R^2_{Y\sim U\mid \tilde{X},\dot{X}_I,Z} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\mid \tilde{X},\dot{X}_I,Z}$
	$R^2_{Y\sim U\mid \tilde{X},\dot{X}_I,Z,D} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\mid \tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	$R_{Y\sim Z\mid X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Z\sim U\mid \tilde{X},\dot{X}_{-j}} \leq b_{UZ}\, R^2_{Z\sim \dot{X}_j\mid \tilde{X},\dot{X}_{-j}}$
$Z \to Y$	$R_{Y\sim Z\mid X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Y\sim Z\mid X,U,D} \leq b_{ZY}\, R^2_{Y\sim \dot{X}_j\mid \tilde{X},\dot{X}_{-j},Z,U,D}$

$\}+(\star)$



The bounds can be combined in any way.

Cinelli & Hazlett (2020)

Computation

Computation

- Properties of optimization problem: monotone objective, non-convex constraints, low-dimensional

Computation

- Properties of optimization problem: monotone objective, non-convex constraints, low-dimensional
- Proposal: Tailored **grid-search algorithm**
  - Key idea: using monotonicity to reduce dimensionality
  - Only OLS constraints: $\mathcal{O}(N)$; Any constraints: $\mathcal{O}(N^3)$ $N =$ #grid points

Overview

1. Applied Example: NLSYM data
2. General Framework
3. Sensitivity Analysis for Regression and IV models
4. **R-package `optsens`**
5. Discussion

Effect of Education on Income

In this vignette we use the `optsens` package to conduct sensitivity analysis for a regression and instrumental variable (IV) model that estimates the linear causal effect of education on income. This package is based on Freidling and Zhao (2025).

The NLSYM Data

The National Longitudinal Survey of Young Men (NLSYM) was initiated in the United States in 1966 and contains a sample of 3010 young men between the ages of 14 and 24 who were followed up until 1981. This data set was compiled and analysed by Card (1993) who used it to investigated the causal effect of education on log-income. The data is available in the R-package ivmodel. We load the data and select the variables that we use in the analysis below.

```
library(ivmodel)

data(card.data)
data <- card.data[,c("lwage", "educ", "nearc4",
                     "exper", "expersq", "black", "south", "smsa")]
knitr::kable(head(data))
```

lwage	educ	nearc4	exper	expersq	black	south	smsa
6.306	7	0	16	256	1	0	1
6.176	12	0	9	81	0	0	1

lwage	educ	nearc4	exper	expersq	black	south	smsa
6.581	12	0	16	256	0	0	1
5.521	11	1	10	100	0	0	1
6.592	12	1	16	256	0	0	1
6.215	12	1	8	64	0	0	1

Description of the variables:

- `lwage`: log-transformed income
- `educ`: education measured in years of schooling
- `nearc4`: indicator for the presence of a 4-year college in the local labour market
- `exper`: potential experience (age - education - 6)
- `expersq`: quadratic transformation of `exper`
- `black`: indicator for being black
- `south`: indicator for living in the southern United States
- `smsa`: indicator for living in a standard metropolitan statistical area

Regression and IV Estimates

To begin with, we examine two standard approaches of estimating the linear causal effect of education on income: the ordinary least squares (OLS) estimator stemming from a regression model and the two stage least squares (TSLS) estimator stemming from an IV model.

For the regression model, we use the `lm` function.

```
reg.mod <- lm(lwage ~ educ + nearc4 + exper + expersq + black + south + smsa,
              data = data)

print(coef(reg.mod)["educ"])
#>    educ
#> 0.07368
print(confint(reg.mod, level = 0.95)["educ",])
#>   2.5 %  97.5 %
#> 0.06679 0.08058
```

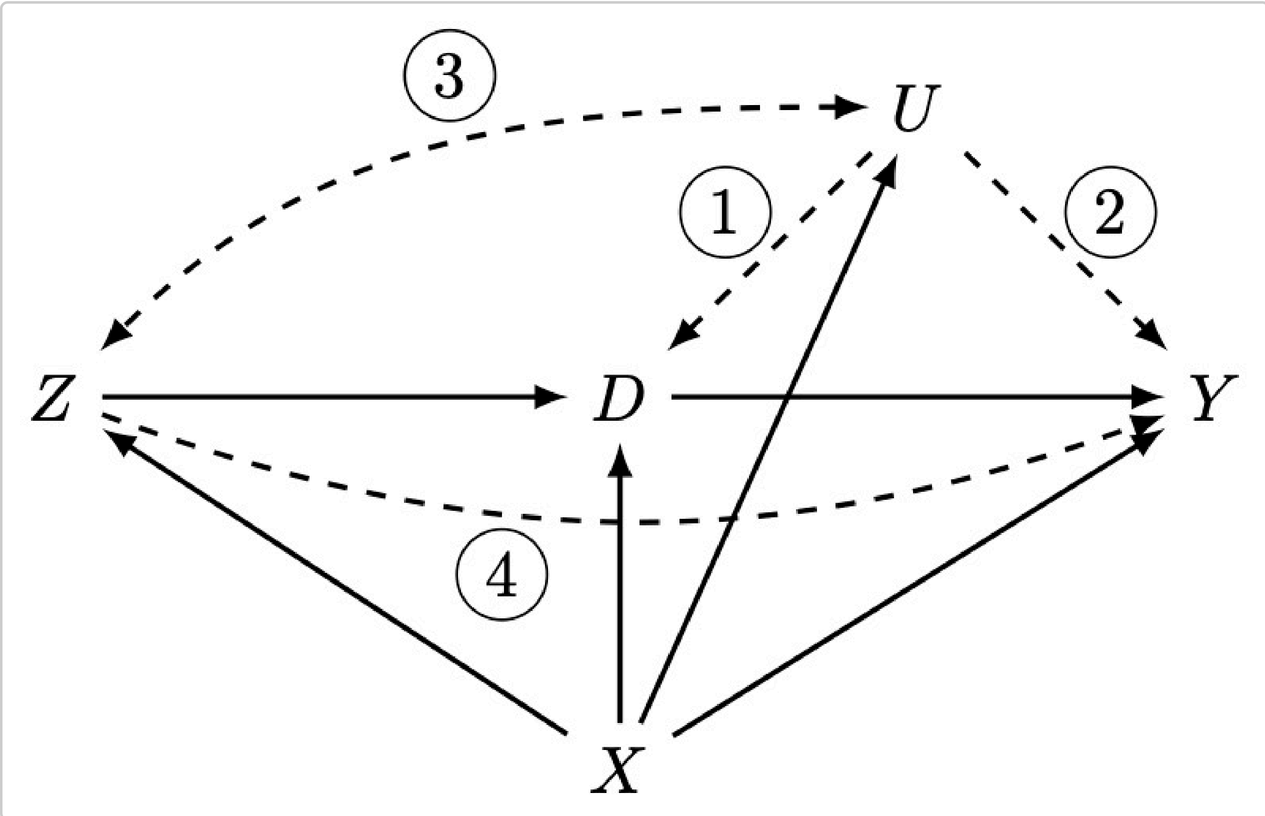
For the IV model, we use the `ivmodel` and `KClass` functions from the `ivmodel` package that we have loaded above.

```
y <- data[, "lwage"]
d <- data[, "educ"]
z <- data[, "nearc4"]
x <- data[, c("exper", "expersq", "black", "south", "smsa")]

iv.mod <- ivmodel(Y = y, D = d, X = x, Z = z,
                  intercept = TRUE, alpha = 0.05, k = 1)
kclass <- KClass(iv.mod, k = 1, alpha = 0.05)
print(kclass$point.est)
#>   Estimate
#> 1   0.1323
print(kclass$ci)
#>     2.5 % 97.5 %
#> 1 0.03575 0.2288
```

We see that the two estimates are quite different despite trying to estimate the same quantity. Moreover, the regression model yields a very narrow 95% confidence interval whereas the IV model provides a comparatively wide one.

Indeed, one needs to be careful under which conditions the OLS and TSLS estimates actually warrant a causal interpretation.

First, we need to make some assumptions on the “order” of variables in the data-generating mechanism. (This is important for choosing the right adjustment set, for instance.) For the NLSYM data, we assume that the variables accord to the directed acyclic graph (DAG) below. This is plausible as the variables are measured in different years which naturally imposes a temporal order.



Second, we require assumptions on variables that are not observed and therefore not part of the data set. Note that we develop our methodology for *one* unmeasured confounder U but one can also think about U as a "super-confounder" that combines the influence of multiple unmeasured variables. The OLS estimator is unbiased when one of the following two conditions holds

$$R_{D\sim U|X,Z} = 0, \qquad R_{Y\sim U|X,Z,D} = 0,$$

where $R_{A\sim B|C}$ denotes the partial correlation of A and B given C. In the graph, this corresponds to the absence of at least one of the edges 1 and 2. The TSLS estimator is unbiased when the following three conditions hold

$$R_{D\sim Z|X} \neq 0, \quad R_{Z\sim U|X} = 0, \quad R_{Y\sim Z|X,U,D} = 0.$$

The first condition corresponds to the existence of the edge $Z \to D$, whereas the latter two correspond to the absence of the edges 3 and 4, respectively.

As Card (1993) pointed out, it is likely that there are important unmeasured variables U that render the identification assumptions for the OLS and TSLS estimators above invalid. Hence, we recommend to conduct *sensitivity analysis* that specifies a range of plausible values for the sensitivity parameters $R_{D\sim U|X,Z}$, $R_{Y\sim U|X,Z,D}$, $R_{Z\sim U|X}$ and $R_{Y\sim Z|X,U,D}$, instead of overly optimistically assuming that they are equal to 0.

Partially Identified Range and Sensitivity Intervals

To conduct sensitivity analysis for the OLS and TSLS estimator, we first load the `optsens` package. Then, we generate a sensitivity analysis object `sa` with the NLSYM data. (The parameters `indep_x` and `dep_x` describe two disjoint subsets of the covariates. This is explained in the subsection on comparative bounds in more detail.) Then, we print the newly created sensitivity object to inspect it.

```
library(optsens)
```

```
sa <- sensana(y = y, d = d, indep_x = c("black", "south"),
              dep_x = c("exper", "expersq", "smsa"),
              quantile = "t", x = x, z = z, alpha = 0.05)

print(sa, digits = 5)
#> Sensitivity Analysis:
#>
#> Dependent Covariates:  exper expersq smsa
#> Independent Covariates:  black south
#>
#> Estimators:
#> OLS   0.07368
#> TSLS  0.1323
#>
#> 95% Confidence Intervals:
#> OLS  [ 0.06679 , 0.08058 ]
#> TSLS [ 0.03575 , 0.2288 ]
#>
#> Specified Bounds:
#> [1] arrow kind  lb    ub    b     I     J
#> <0 rows> (or 0-length row.names)
```

We see that sa contains the OLS and TSLS estimator along with their 95% confidence intervals. These agree with the outputs of lm and ivmodel. Moreover, sa administers a (currently empty) data frame containing bounds on the sensitivity parameters.

In the following two subsections, we describe how to add different types of bounds and compute the partially identified range as well as sensitivity intervals. For ease of exposition, we assume that the unmeasured confounder U is the intrinsic motivation of a person but one may think of various other unmeasured variables and conduct sensitivity analysis for them in a similar way.

Direct Bounds

The most straightforward way of specifying a sensitivity model is putting direct bounds on the sensitivity parameters. For instance, we may impose

$$R_{D\sim U|X,Z} \in [-0.2, 0.5], \qquad R_{Y\sim U|X,Z,D} \in [-0.2, 0.4].$$

The first constraint expresses the belief that the partial correlation of education and motivation given the covariates and the instrument lies in between -0.2 and 0.5. Since a positive dependence between education and motivation seems more plausible, the interval gives more leeway in the positive direction. We may apply a similar reasoning for the partial correlation between income and motivation and the second bound. We can add these constraints to `sa` as follows.

```
sa <- add_bound(sa, arrow = "UD", kind = "direct", lb = -0.2, ub = 0.5)
sa <- add_bound(sa, arrow = "UY", kind = "direct", lb = -0.2, ub = 0.4)
```

We see that the bounds were indeed successfully added and given the names `b1` and `b2`. (The names can also be customized in the `add_bound` function.)

```
print(sa)
#> Sensitivity Analysis:
#>
#> Dependent Covariates:  exper expersq smsa
#> Independent Covariates:  black south
#>
#> Estimators:
#> OLS   0.074
#> TSLS  0.132
```

```
#>
#> 95% Confidence Intervals:
#> OLS  [ 0.067 , 0.081 ]
#> TSLS [ 0.036 , 0.229 ]
#>
#> Specified Bounds:
#>    arrow   kind   lb  ub  b    I    J
#> b1    UD direct -0.2 0.5 NA NULL NULL
#> b2    UY direct -0.2 0.4 NA NULL NULL
```

Subject to the specified bounds, the linear causal effect of education on income is not point identified anymore. This means that we can only estimate a (partially identified) range of values instead of a single number. The `optsens` package uses a grid search algorithm to this end. (We recommend to use at least 100 grid points per dimension.)

```
grid_specs <- list(N1 = 200, N2 = 200, N5 = 200)
pir1 <- pir(sa, grid_specs = grid_specs)

print(pir1)
#> [1] 0.02920 0.09593
```

The *partially identified range* (PIR) that we obtain of course contains the OLS estimate but does not contain 0. Hence, we can conclude that the estimate of the causal effect is still positive even if unmeasured confounding occurs that is no larger than the specified bounds.

The PIR, however, does not account for sampling variability. Therefore, we also want to construct a confidence interval for the PIR which we call a *sensitivity interval*. To this end, we use a bootstrap approach which is implemented in the `sensint` function.

```
sensint1 <- sensint(sa, alpha = 0.05, boot_samples = 3500,
                    grid_specs = grid_specs)
print(sensint1)
#> 95% Sensitivity Intervals:
#>        sl     su bootstrap conservative
#> 1 0.02162 0.1034   percent        FALSE
#> 2 0.02162 0.1034   percent         TRUE
#> 3 0.02151 0.1033     basic        FALSE
#> 4 0.02151 0.1033     basic         TRUE
#> 5 0.02162 0.1036       bca        FALSE
#> 6 0.02162 0.1036       bca         TRUE
```

The resulting data frame contains the lower and upper ends (`sl` and `su`) of the sensitivity interval for different bootstrap procedures (`percent`, `basic` and `bca`). We recommend to mostly use the BCa method as it corrects for bias and skewness in the bootstrap distribution. Note that `bca` requires at least as many bootstrap samples `boot_samples` as there are data points in the data set. The last column `conservative` is only relevant when comparative bounds are specified; see the following subsection. In general, we recommend to use the conservative sensitivity interval.

We notice that the 95% sensitivity interval does not contain 0. Hence, we can conclude that even in the presence of the specified unmeasured confounding a positive effect of education on income is significant.

Beyond bounds on $R_{D\sim U|X,Z}$ and $R_{Y\sim U|X,Z,D}$, we can also put bounds on the TSLS sensitivity parameters.

```
sa <- add_bound(sa, arrow = "ZU", kind = "direct", lb = -0.3, ub = 0.3)
sa <- add_bound(sa, arrow = "ZY", kind = "direct", lb = -0.1, ub = 0.1)
```

Note that bounds on $R_{Z\sim U|X}$ and $R_{Y\sim Z|X,U,D}$ alone are generally not sufficient to get a finite PIR. Hence, we recommend to specify at least one other bound when doing sensitivity analysis for the TSLS assumptions.

We compute the PIR and sensitivity interval of the updated sensitivity model as follows.

```
pir2 <- pir(sa, grid_specs = grid_specs)
print(pir2)
#> [1] 0.02920 0.09593

sensint2 <- sensint(sa, alpha = 0.05, boot_samples = 3500,
                    grid_specs = grid_specs)
print(sensint2)
#> 95% Sensitivity Intervals:
#>         sl     su bootstrap conservative
#> 1 0.02149 0.1033   percent        FALSE
#> 2 0.02149 0.1033   percent         TRUE
#> 3 0.02158 0.1035     basic        FALSE
#> 4 0.02158 0.1035     basic         TRUE
#> 5 0.02172 0.1035       bca        FALSE
#> 6 0.02172 0.1035       bca         TRUE
```

Interestingly, `pir1` and `pir2` are identical and `sensint1` and `sensint2` are almost the same. This indicates that the two new constraints did not contribute to the sensitivity model. This is a first sign that even small violations of the TSLS assumptions may lead to large deviations from the estimate.

To investigate this further, we remove the previous bounds on $U \to D$ and $U \to Y$, put a very loose bound on $U \to D$ and recompute the PIR and sensitivity interval.

```
sa <- remove_bound(sa, "b1")
sa <- remove_bound(sa, "b2")
sa <- add_bound(sa, arrow = "UD", kind = "direct", lb = -0.9, ub = 0.9)
```

```r
pir3 <- pir(sa, grid_specs = grid_specs)
print(pir3)
#> [1] -0.3236  0.4710

sensint3 <- sensint(sa, alpha = 0.05, boot_samples = 3500,
                    grid_specs = grid_specs)
print(sensint3)
#> 95% Sensitivity Intervals:
#>        sl     su bootstrap conservative
#> 1 -0.3413 0.4877   percent        FALSE
#> 2 -0.3413 0.4877   percent         TRUE
#> 3 -0.3397 0.4870     basic        FALSE
#> 4 -0.3397 0.4870     basic         TRUE
#> 5 -0.3416 0.4876       bca        FALSE
#> 6 -0.3416 0.4876       bca         TRUE
```

Both the PIR and the sensitivity interval are quite large compared to the expected effect size. This shows that even moderately small deviations from the TSLS identification assumptions can lead to vastly different estimates and statistical conclusions. Therefore, the regression approach seems a lot more robust towards unmeasured confounding than the IV approach for the NLSYM data.

In many cases, it may be hard to confidently specify a plausible range of values for the sensitivity parameters. Therefore, we introduce bounds that compare U with an observed covariate in the next subsection.

Comparative Bounds

First, we remove the previously specified bounds. This is purely for ease of exposition, however.

```
sa <- remove_bound(sa, "b3")
sa <- remove_bound(sa, "b4")
sa <- remove_bound(sa, "b5")

print(sa)
#> Sensitivity Analysis:
#>
#> Dependent Covariates:  exper expersq smsa
#> Independent Covariates:  black south
#>
#> Estimators:
#> OLS   0.074
#> TSLS  0.132
#>
#> 95% Confidence Intervals:
#> OLS  [ 0.067 , 0.081 ]
#> TSLS [ 0.036 , 0.229 ]
#>
#> Specified Bounds:
#> [1] arrow kind  lb    ub    b     I     J
#> <0 rows> (or 0-length row.names)
```

To specify comparative bounds, we need to make one additional assumption on the covariates: We suppose that X can be divided into two disjoint sets $\widetilde{X}$ such $\dot{X}$ that the following condition holds

$$R^2_{U \sim \dot{X} | \widetilde{X}, Z} = 0.$$

This means that the covariates $\dot{X}$ cannot explain any variation in the unmeasured confounder if we also account for $\widetilde{X}$ and the instrument Z. Therefore, one may think about $\dot{X}$ as covariates that are in some sense "independent" of the unmeasured confounder and of $\widetilde{X}$ as "dependent". When we created the `sa` object at the start of the previous section, we specified this partition of the covariates via the `indep_x` and `dep_x` parameters.

We chose the covariates `black` and `south` as $\dot{X}$-covariates because we believe that motivation U does not correlate with them after controlling for other covariates. For the remaining covariates, we are not entirely sure; so, we designate them as $\widetilde{X}$-covariates. In the following, we use covariates in $\dot{X}$ and compare them to the unmeasured confounder U to obtain more interpretable bounds.

Let us first consider the OLS sensitivity parameters. We may for instance choose the indicator for being black (here denoted as $\dot{X}_j$) as comparison variable and specify the comparative bound

$$R^2_{D\sim U|\widetilde{X},\dot{X}_{-j},Z} \leq 4\, R^2_{D\sim \dot{X}_j|\widetilde{X},\dot{X}_{-j},Z} .$$

This bound expresses the belief that being black can explain *at most 4 times* as much variation in education as motivation can, after controlling for the other covariates and the instrument. In many settings, it may be easier to specify such a comparative bound and reason about whether one should choose a larger or smaller number than 4 than imposing a direct bound on $R_{D\sim U|X,Z}$.

Similarly, we can also specify a bound on the relationship between Y and U:

$$R^2_{Y\sim U|\widetilde{X},\dot{X}_{-j},Z,D} \leq 5\, R^2_{Y\sim \dot{X}_j|\widetilde{X},\dot{X}_{-j},Z,D} .$$

This corresponds to the belief that being black can explain *at most 5 times* as much variation in income than motivation can, after accounting for the remaining covariates, the instrument and education D.

We can add these constraints to `sa` as follows.

```
sa <- add_bound(sa, arrow = "UD", kind = "comparative", b = 4, I = "south", J = "black")
sa <- add_bound(sa, arrow = "UY", kind = "comparative-d", b = 5, I = "south", J = "black")
```

```
print(sa)
#> Sensitivity Analysis:
#>
#> Dependent Covariates:  exper expersq smsa
#> Independent Covariates:  black south
#>
#> Estimators:
#> OLS   0.074
#> TSLS  0.132
#>
#> 95% Confidence Intervals:
#> OLS  [ 0.067 , 0.081 ]
#> TSLS [ 0.036 , 0.229 ]
#>
#> Specified Bounds:
#>    arrow          kind     lb    ub b     I     J
#> b6    UD   comparative -0.410 0.410 4 south black
#> b7    UY comparative-d -0.431 0.431 5 south black
```

Here, `J` refers to the comparison covariate and `I` is a subset (usually the entire set) of the remaining $\dot{X}$-covariates. There also exists a version of the second bound that does not control for D; to apply this constraint, use `kind = "comparative"`.

As before, we can now compute the PIR and the sensitivity interval.

```
pir4 <- pir(sa, grid_specs = grid_specs)
print(pir4)
#> [1] 0.04367 0.13341
```

```
sensint4 <- sensint(sa, alpha = 0.05, boot_samples = 3500,
                    grid_specs = grid_specs)
print(sensint4)
#> 95% Sensitivity Intervals:
#>         sl     su bootstrap conservative
#> 1 0.02998 0.1575   percent        FALSE
#> 2 0.02998 0.1575   percent         TRUE
#> 3 0.03156 0.1528     basic        FALSE
#> 4 0.03156 0.1528     basic         TRUE
#> 5 0.02979 0.1567       bca        FALSE
#> 6 0.02979 0.1567       bca         TRUE
```

Similarly, we can also specify comparative constraints for the TSLS sensitivity parameters, for instance:

$$
\begin{aligned}
R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}} &\leq 0.5\, R^2_{Z\sim \dot{X}_j|\tilde{X},\dot{X}_{-j}} \\
R^2_{Y\sim Z|X,U,D} &\leq 0.1\, R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z,U,D}
\end{aligned}
$$

We can add these to `sa` as follows. (Note that we do not need to specify `I` here as this is automatically set as the remaining $\dot{X}$-covariates)

```
sa <- add_bound(sa, arrow = "ZU", kind = "comparative", b = 0.5, J = "black")
sa <- add_bound(sa, arrow = "ZY", kind = "comparative", b = 0.1, J = "black")
```

We recompute the partially identified range and the 95% sensitivity interval.

```r
pir5 <- pir(sa, grid_specs = grid_specs)
print(pir5)
#> [1] 0.04367 0.13341

sensint5 <- sensint(sa, alpha = 0.05, boot_samples = 3500,
                    grid_specs = grid_specs)
print(sensint5)
#> 95% Sensitivity Intervals:
#>        sl     su bootstrap conservative
#> 1 0.02983 0.1577   percent        FALSE
#> 2 0.02954 0.1580   percent         TRUE
#> 3 0.03071 0.1532     basic        FALSE
#> 4 0.03039 0.1538     basic         TRUE
#> 5 0.02965 0.1578       bca        FALSE
#> 6 0.02944 0.1582       bca         TRUE
```

Analogously to the sensitivity model with direct bounds, we notice that adding constraints on $U \leftrightarrow Z$ and $Z \rightarrow Y$ does not palpably improve neither the PIR nor the sensitivity intervals. Hence, the analysis is much more sensitive to violations of the TSLS than the OLS assumptions.

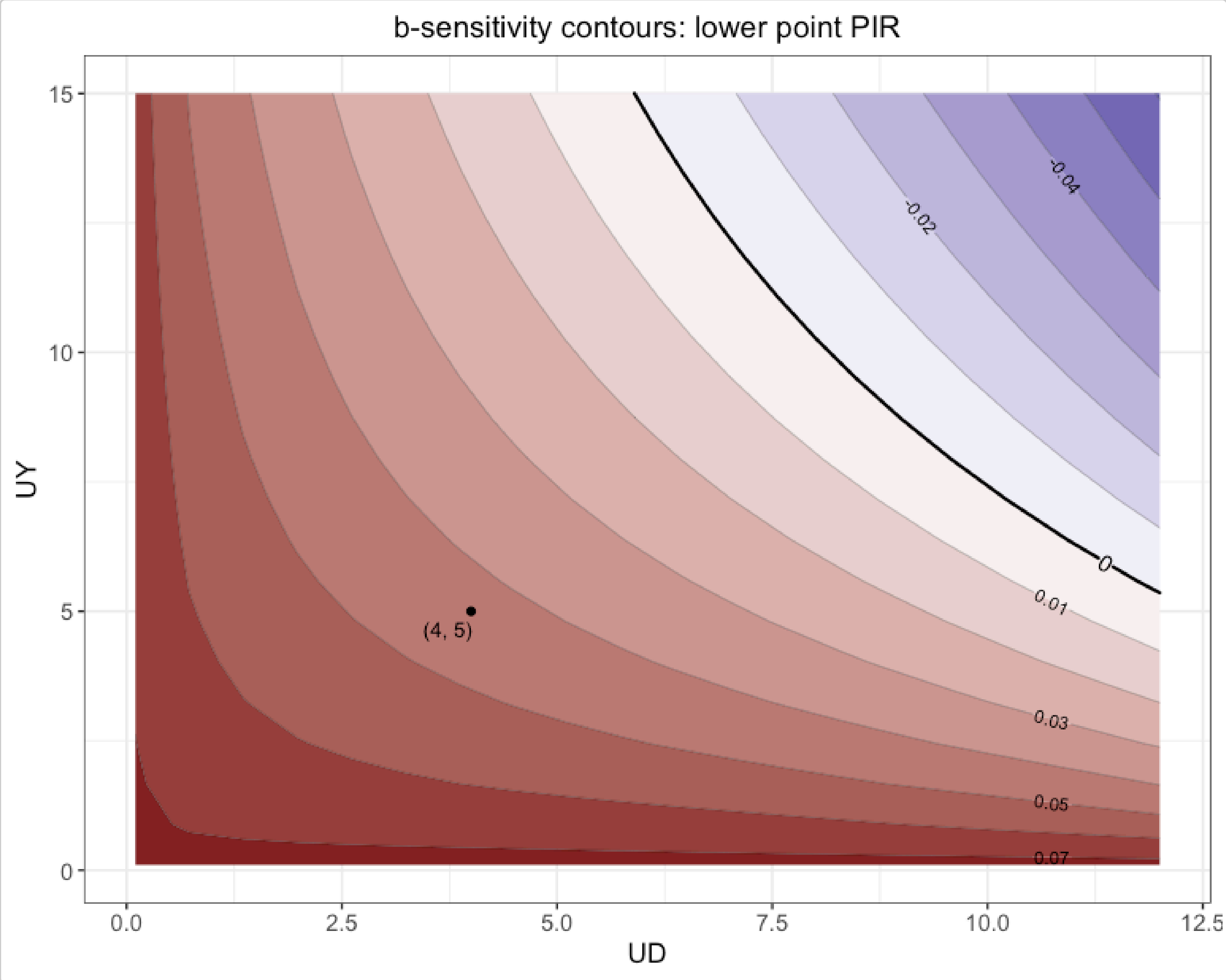
We'd like to emphasize that users can use both direct and indirect bounds including on the same arrow. In this vignette, we have only separated them to explain them more easily.

b- and R-contour plots

When we specify comparative bounds, we need to choose the comparison covariate and the b-factor, e.g. being black explains at most 4 times as much variation as motivation, here $b = 4$. To examine multiple choices of b at the same time, we can use the `b_contours` function.

We choose two comparative bounds ("b6" and "b7" in this example) and two corresponding ranges of b-values and specify whether we want to plot the lower or upper end of the PIR. Moreover, we choose a value of interest (usually 0) and the grid_specs_b parameter which controls how finely range1 and range2 are discretized. b_contours produces a ggplot contour plot that can be printed.

```
b_contour_plot <- b_contours(sa, pir_lower = TRUE,
                             bound1 = "b6", range1 = c(0.1, 12),
                             bound2 = "b7", range2 = c(0.1, 15),
                             val_interest = 0,
                             grid_specs_b = list(N1b = 20, N2b = 20),
                             grid_specs = grid_specs)

print(b_contour_plot)
```



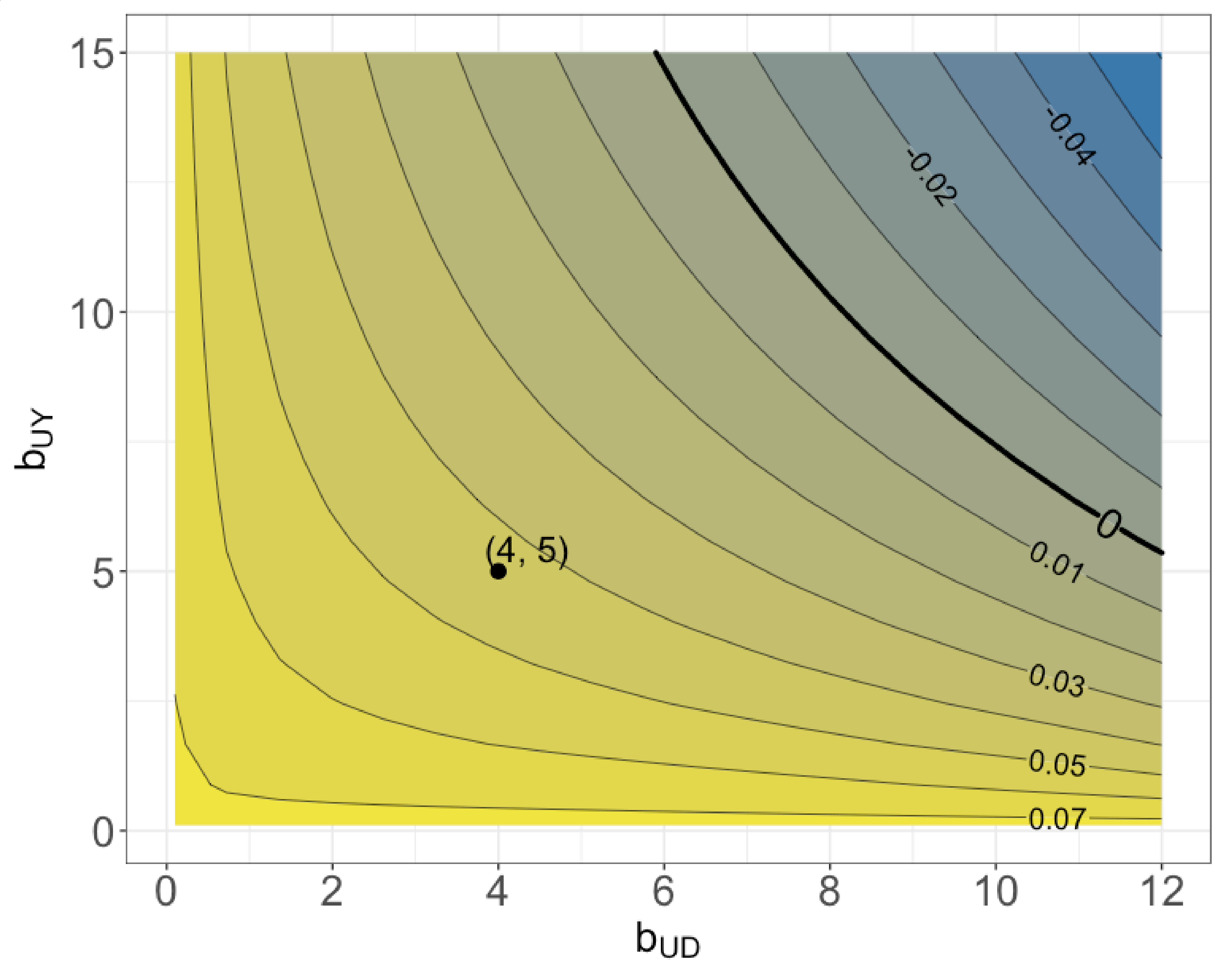
Bounds "b6" and "b7" use b-factors of 4 and 5, respectively. The plot shows that we need to choose values at least twice as large to push the the lower end of the PIR beyond zero. Hence, even more permissive sensitivity models can lead to the qualitatively same conclusion that the estimated effect is positive even when unmeasured confounding occurs.

If users would like to customize the plot, they can use the `b_contours_data` function to just create the data frame which the contour plot is based on and then visualize it themselves. Here is an example how one of the figures in Freidling and Zhao (2025) is created.

```
library(ggplot2)
library(ggrepel)
library(metR)

plot_data <- b_contours_data(sa, pir_lower = TRUE,
                             bound1 = "b6", range1 = c(0.1, 12),
                             bound2 = "b7", range2 = c(0.1, 15),
                             grid_specs_b = list(N1b = 20, N2b = 20),
                             grid_specs = grid_specs)

print(head(plot_data))
#>        x   y       z
#> 1 0.1000 0.1 0.07320
#> 2 0.7263 0.1 0.07314
#> 3 1.3526 0.1 0.07314
#> 4 1.9789 0.1 0.07314
#> 5 2.6053 0.1 0.07314
#> 6 3.2316 0.1 0.07314


text_point <- paste0("(", 4, ", ", 5, ")")
make_breaks <- function(range, binwidth) {
  signif(pretty(range, 15), 4)
```

```
}
make_breaks_ex <- function(range, binwidth) {
  b <- make_breaks(range, binwidth)
  b[b != 0]
}

pl <- ggplot(plot_data, aes(x, y, na.rm = TRUE)) +
  geom_contour_fill(aes(z = z, fill = after_stat(level)),
                    breaks = make_breaks,
                    show.legend = FALSE) +
  geom_contour2(aes(z = z, label = after_stat(level)),
                breaks = make_breaks_ex,
                col = "black",
                label_size = 5,
                linewidth = 0.25) +
  geom_contour2(aes(z = z, label = after_stat(level)),
                breaks = 0,
                linewidth = 1.25,
                label_size = 7,
                col = "black") +
  scale_fill_discretised(low = "#0072B2", high = "#F0E442") +
  geom_point(data = data.frame(x = 4, y = 5), size = 3,
             mapping = aes(x, y), col = "black") +
  geom_text_repel(data = data.frame(x = 4, y = 5, label = text_point),
                  mapping = aes(x, y, label = label),
                  col = "black",
                  size = 6,
                  xlim = c(0, 13),
                  ylim = c(0, 15)) +
```

```
    labs(x = expression("b"["UD"]),
         y = expression("b"["UY"])) +
    scale_x_continuous(breaks = seq(0, 12, by = 2)) +
    theme_bw() +
    theme(plot.title = element_blank(),
          axis.text = element_text(size = 20),
          axis.title = element_text(size = 20),
          title = element_text(size = 20))

print(pl)
```



Beyond b-contour plots, we offer R-contour plots which visualize the estimated causal effect as a function of $R_{D\sim U|X,Z}$ and $R_{Y\sim U|X,Z,D}$. As this does not require bounds, we can remove the previously specified bounds.

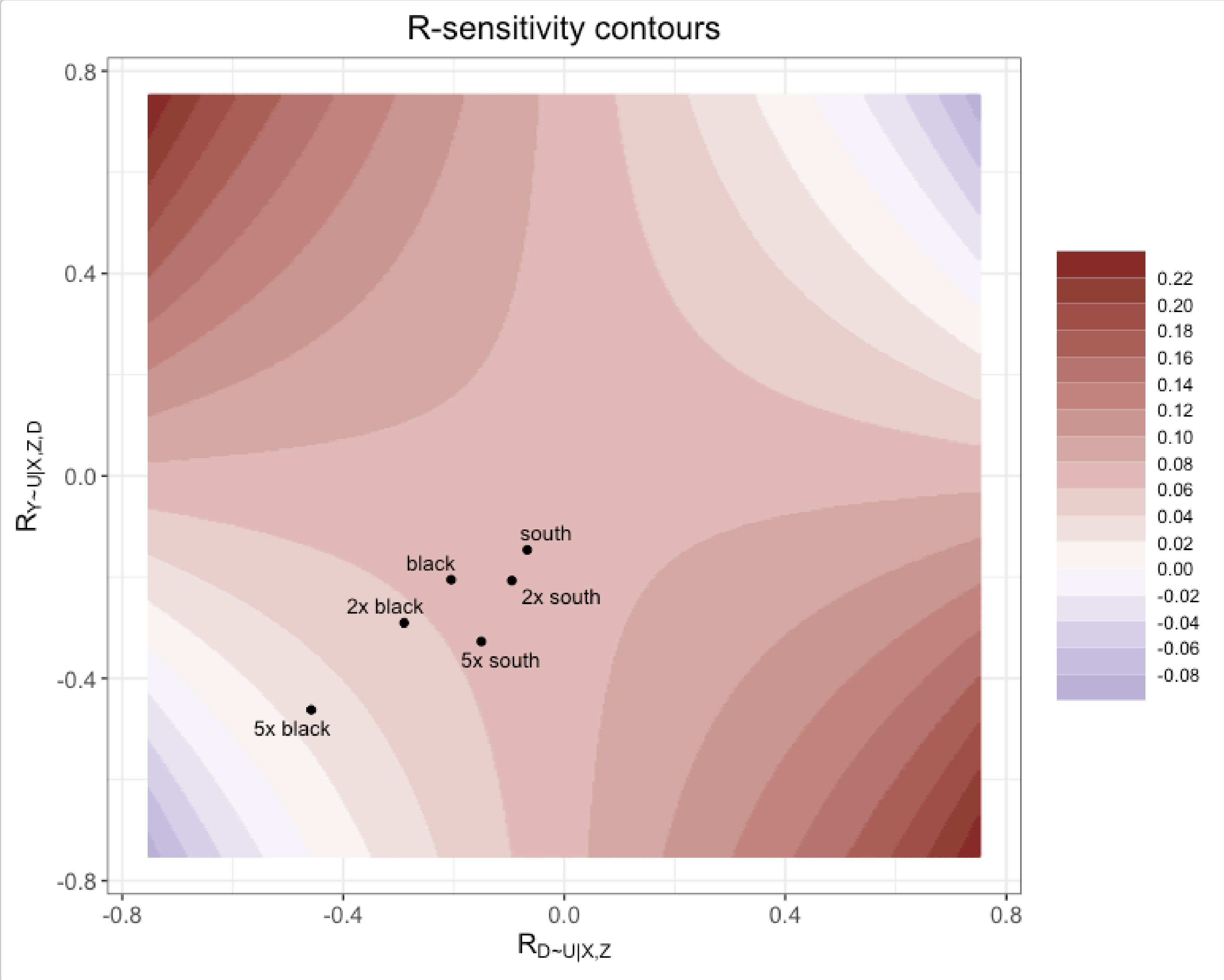
In order to focus on the non-extreme values of the sensitivity parameters - in other words *to zoom in* - we specify direct bounds on $U \to D$ and $U \to Y$. This is optional, though.

```
sa <- remove_bound(sa, "b6")
sa <- remove_bound(sa, "b7")
sa <- remove_bound(sa, "b8")
sa <- remove_bound(sa, "b9")

sa <- add_bound(sa, arrow = "UD", kind = "direct", lb = -0.75, ub = 0.75)
sa <- add_bound(sa, arrow = "UY", kind = "direct", lb = -0.75, ub = 0.75)
```

In addition to the contours of the estimated causal effect, we can plot comparison points that provide context for the order of magnitude of the sensitivity parameters. To this end, we choose some of the $\dot{X}$-covariates and different multipliers for each of them. This is done in the `comparison_ind` list. We pass this parameter along with other already familar ones to `r_contours`. Here, we choose `comparison = "comp-d"`; other options are `"comp"` (analogous to a comparative bound on $U \to Y$ that does not account for D) and `"naive"` which is not recommended. Setting `iv_lines = TRUE` would also plot the combinations of values that $R_{D\sim U|X,Z}$ and $R_{Y\sim U|X,Z,D}$ need to fulfill if Z is indeed a valid instrument; these combinations typically form two lines in the plotted plane.

```
comparison_ind <- list(black = c(1, 2, 5), south = c(1, 2, 5))
grid_specs <- list(N1 = 400, N2 = 400, N5 = 400)

r_contour_plot <- r_contours(sa, val_interest = 0,
                             comparison_ind = comparison_ind,
                             comparison = "comp-d",
                             iv_lines = FALSE,
                             grid_specs = grid_specs)
```

```r
print(r_contour_plot)
```



The R-contour plot shows that an unmeasured confounder that is five times as strong as being black or living in the southern US would still not suffice to completely push the estimate to 0.

Analogously to the b-contour plots, there also exists an `r_contours_data` function which provides the underlying data and allows the users to create the plot themselves according to their preferences. See the Github repository optsens-replication for more examples of how to use this and other functions.

References

Card, David. 1993. "Using Geographic Variation in College Proximity to Estimate the Return to Schooling." w4483. National Bureau of Economic Research.

Freidling, Tobias, and Qingyuan Zhao. 2025. "Optimization-Based Sensitivity Analysis for Unmeasured Confounding Using Partial Correlations." *arXiv:2301.00040*.

Overview

1. Applied Example: NLSYM data
2. General Framework
3. Sensitivity Analysis for Regression and IV models
4. R-package `optsens`
5. **Discussion**

Discussion

Discussion

- Optimization-based Sensitivity Analysis: **more interpretable and complex** sensitivity models possible
- Estimation of PIR via plug-in approach; sensitivity intervals via bootstrap
- Efficient optimization algorithm and package
- Visualization tools: b- and R-contour plots

Discussion

- Optimization-based Sensitivity Analysis: **more interpretable and complex** sensitivity models possible
- Estimation of PIR via plug-in approach; sensitivity intervals via bootstrap
- Efficient optimization algorithm and package
- Visualization tools: b- and R-contour plots

- Future research:
  - Theoretical guarantees, esp. bootstrap theory
  - Application to more settings

Thanks for your attention!

Paper

R-package

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References

Balke, A. and Pearl, J. (1994) ‘Counterfactual Probabilities: Computational Methods, Bounds and Applications’, in Uncertainty Proceedings 1994. San Francisco (CA): Morgan Kaufmann, pp. 46–54.

Balke, A. and Pearl, J. (1997) ‘Bounds on Treatment Effects from Studies with Imperfect Compliance’, Journal of the American Statistical Association, 92(439), pp. 1171–1176.

Card, D. (1993) *Using Geographic Variation in College Proximity to Estimate the Return to Schooling*. w4483. Cambridge, MA: National Bureau of Economic Research, p. w4483.

Cinelli, C. and Hazlett, C. (2020) ‘Making sense of sensitivity: extending omitted variable bias’, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 82(1), pp. 39–67.

Ding, P. and VanderWeele, T.J. (2016) ‘Sensitivity Analysis Without Assumptions’, Epidemiology (Cambridge, Mass.), 27(3), pp. 368–377.

Freidling, T. and Zhao, Q. (2025) ‘Optimization-based Sensitivity Analysis for Unmeasured Confounding using Partial Correlations’. arXiv.2301.00040

Imbens, G.W. and Manski, C.F. (2004) ‘Confidence Intervals for Partially Identified Parameters’, Econometrica, 72(6), pp. 1845–1857.

Rosenbaum, P.R. (1987) ‘Sensitivity analysis for certain permutation inferences in matched observational studies’, Biometrika, 74(1), pp. 13–26.

R^2-calculus

(i) *Independence*: If $Y \perp\!\!\!\perp X$, then $R^2_{Y\sim X} = 0$.

(ii) *Independent additivity*: If $X \perp\!\!\!\perp W$, then $R^2_{Y\sim X+W} = R^2_{Y\sim X} + R^2_{Y\sim W}$.

(iii) *Decomposition of unexplained variance*: $1 - R^2_{Y\sim X+W} = (1 - R^2_{Y\sim X})(1 - R^2_{Y\sim W|X})$

(iv) *Recursion of partial correlation*:

$$R_{Y\sim X|W} = \frac{R_{Y\sim X} - R_{Y\sim W}R_{X\sim W}}{\sqrt{1 - R^2_{Y\sim W}}\sqrt{1 - R^2_{X\sim W}}}$$

(v) *Reduction of partial correlation*: If X and W is one-dimensional and $Y \perp\!\!\!\perp W$, then

$$R_{Y\sim X|W} = \frac{R_{Y\sim X}}{\sqrt{1 - R^2_{X\sim W}}}$$

(vi) *Three-variable restriction*: If X and W is one-dimensional, then

$$f_{Y\sim X|W}\sqrt{1 - R^2_{Y\sim W|X}} = f_{Y\sim X}\sqrt{1 - R^2_{X\sim W}} - R_{Y\sim W|X}R_{X\sim W}.$$

All statements are also true when Z is partialed out, i.e. add "$\,|\, Z$".