Sensitivity Analysis with the R^2-Calculus

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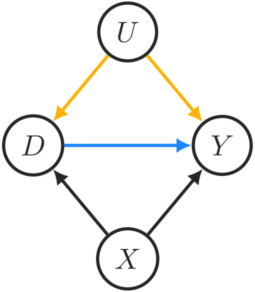
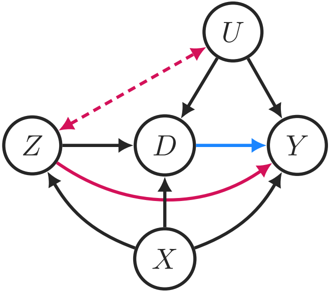
Untestable Assumptions

Regression



There is no unmeasured confounder U, i.e. U cannot effect D and Y (yellow arrows) simultaneously.

Instrumental Variables



The instrument Z influences Y only through D and it is independent of U, that is absence of the red arrows.

R^2-Calculus

In a linear regression $Y = X\beta + \varepsilon$, $R^2_{Y\sim X}$ is the proportion of variance in Y that is explained by the model.

R^2-Calculus

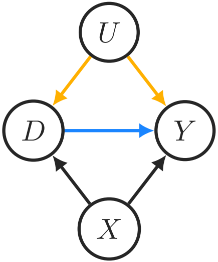
Let $Y \in \mathbb{R}$, $X \in \mathbb{R}^d$, $Z \in \mathbb{R}^k$ and $W \in \mathbb{R}^l$ be random vectors

- $R^2_{Y\sim X} = 1 - \frac{\mathrm{var}(Y - X\beta)}{\mathrm{var}(Y)}$, where β is the regression coefficient
- $R^2_{Y\sim X|Z} = \frac{R^2_{Y\sim X+Z} - R^2_{Y\sim Z}}{1 - R^2_{Y\sim Z}}$
- $\frac{\mathrm{var}(Y^{\perp X,Z})}{\mathrm{var}(Y^{\perp Z})} = 1 - R^2_{Y\sim X|Z}$
- $R_{Y\sim X|Z} = \mathrm{corr}(Y^{\perp Z}, X^{\perp Z})$, for $X \in \mathbb{R}$
- $R_{Y\sim X|Z,W} = \frac{R_{Y\sim X|Z} - R_{Y\sim W|Z} R_{X\sim W|Z}}{\sqrt{1 - R^2_{Y\sim W|Z}}\sqrt{1 - R^2_{X\sim W|Z}}}$, for $X, W \in \mathbb{R}$.
- $f^2_{Y\sim X|Z} = \frac{R^2_{Y\sim X|Z}}{1 - R^2_{Y\sim X|Z}}$; $\quad f_{Y\sim X|Z} = \frac{R_{Y\sim X|Z}}{\sqrt{1 - R^2_{Y\sim X|Z}}}$, for $X \in \mathbb{R}$

Sensitivity Analysis - Linear Regression

Linear regression model:

$$Y = D\beta + U\gamma + \lambda^T X + \varepsilon$$



Bias in the β-estimate when excluding U:

$$\text{bias} = R_{Y\sim U|D,X}\, f_{D\sim U|X}\, \frac{\text{sd}(Y^{\perp D,X})}{\text{sd}(D^{\perp X})}$$

We can find a range for the bias by reasoning about $R_{Y\sim U|D,X}$ and $f_{D\sim U|X}$. For instance, if a researcher believes $R^2_{D\sim U} \leq 0.5\, R^2_{D\sim X}$, we apply the rules of the R^2-calculus and find the bound

$$|f_{D\sim U|X}| \leq \sqrt{\frac{0.5 f^2_{D\sim X}}{1 - 0.5 f^2_{D\sim X}}}.$$

What's more

- Sensitivity analysis for Instrumental Variables
- Multiple instruments
- K-class estimators
- Computer algebra system for the R^2-calculus?

References

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