Sensitivity Analysis with the R^2-Calculus

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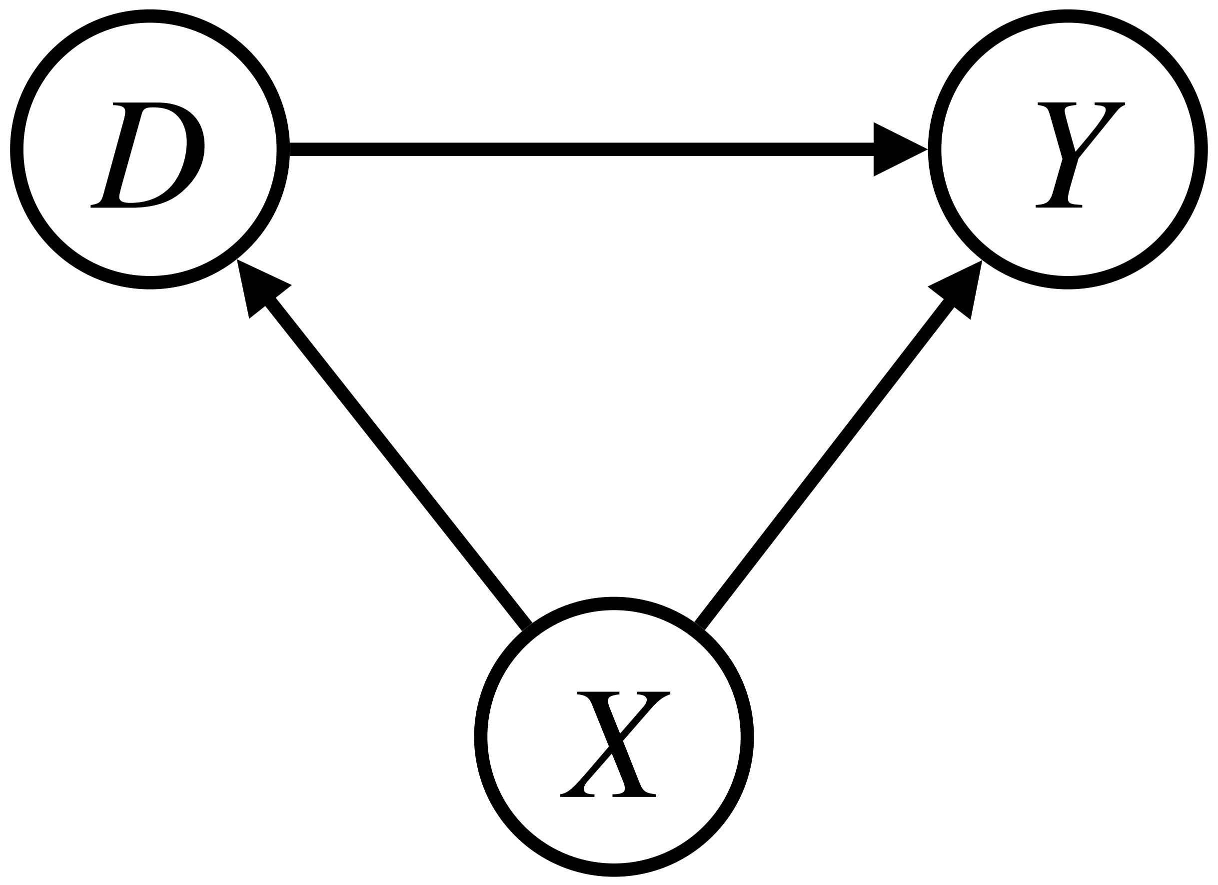
Causal Inference and DAGs

- Directed Acyclic Graph (DAG)
- Linear Structural Equation Model

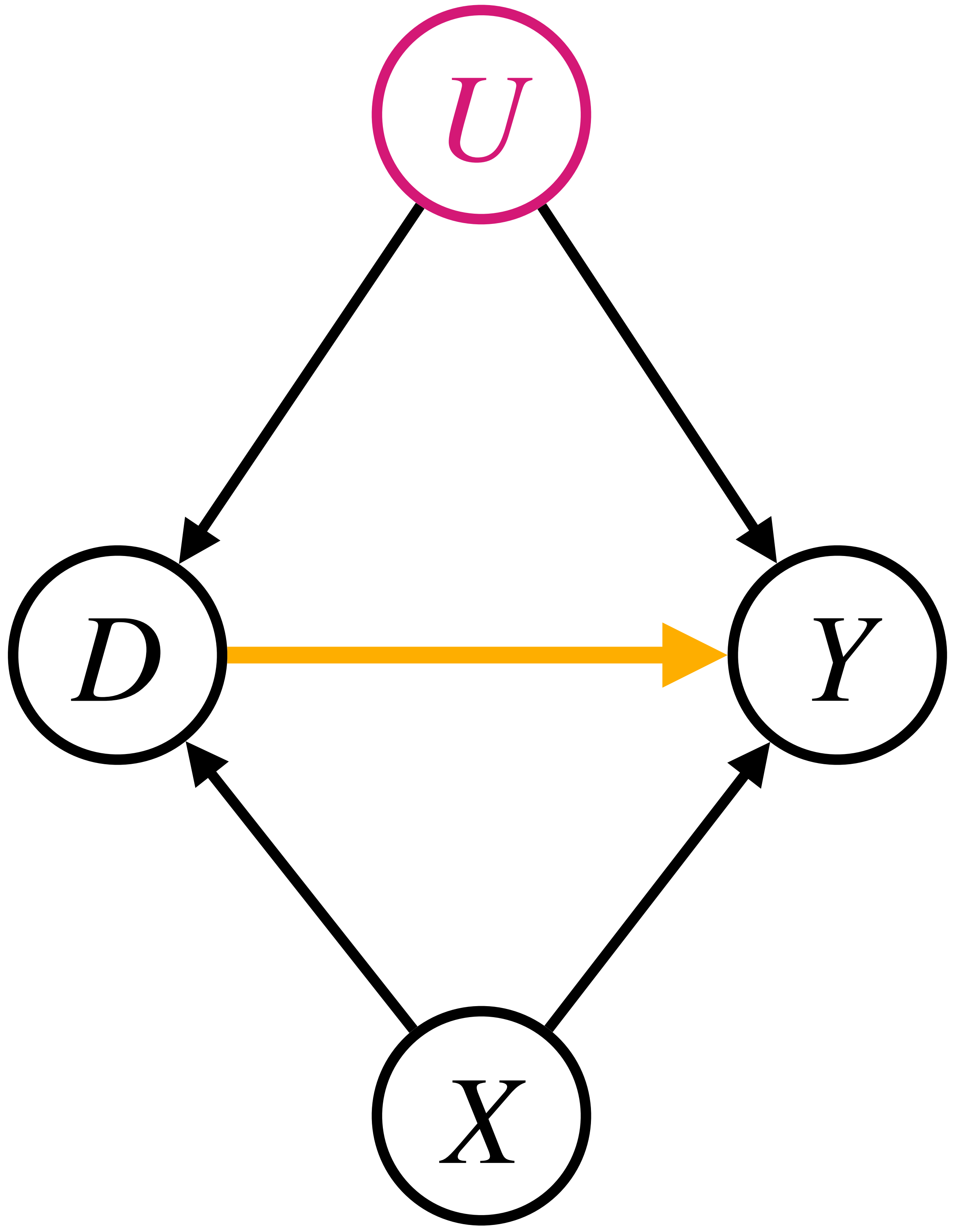
$$X \Leftarrow \varepsilon_X$$
$$D \Leftarrow \beta_{DX} X + \varepsilon_D$$
$$Y \Leftarrow \beta_{YD} D + \beta_{YX} X + \varepsilon_Y$$



- Goal: Estimate the causal effect of D on Y, i.e. estimate β_{YD} from data $(X_i, D_i, Y_i)_{i=1}^n$

Causal Inference and Sensitivity Analysis

- Goal: Estimate β_{YD}
- Assumptions:
  - Correction model specification, i.e. linearity
  - Correct DAG
  - No unmeasured confounders
- Sensitivity analysis explores how violations of assumptions affect estimation
- This work: focus on unmeasured confounders



Sensitivity Analysis - History

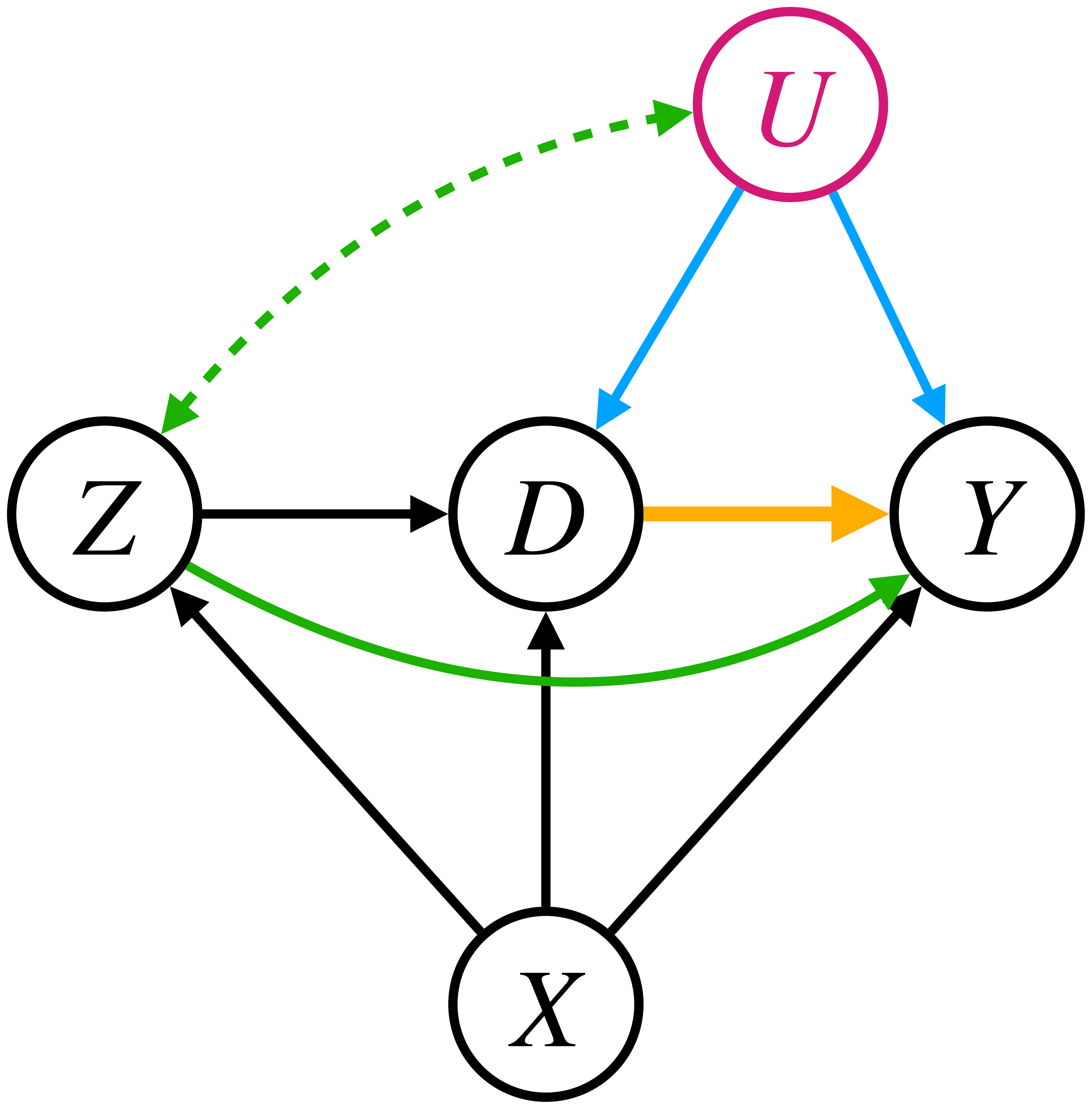
- Cornfield et al. (1959): Association between smoking and lung cancer
- Rosenbaum sensitivity model (Rosenbaum 1987), E-values (Ding and VanderWeele 2016), Marginal sensitivity model (Zhao et al. 2019, Dorn and Guo 2022), Instrumental variables (Altonji et al. 2005, Small 2007) and many more
- Lots of methods but **hardly used in practice**
- Reason: **simplistic** and **unintuitive**
- Proposal: **constrained optimization** $\rightarrow$ more flexibility and interpretability

Sensitivity Analysis - New Framework

- $(V_i, U_i)_{i=1}^n \sim \mathbb{P}_{V,U}$, but only $(V_i)_{i=1}^n$ observed
- Objective: $\beta(\theta, \psi)$
- θ are estimable parameters, i.e. only depend on $\mathbb{P}_V$
- ψ are sensitivity parameters, i.e. depend on $\mathbb{P}_{V,U}$
- Specify domain knowledge as constraints: $g(\theta, \psi) \leq 0$, $h(\theta, \psi) = 0$

$$\min_{\psi} / \max_{\psi} \beta(\hat{\theta}, \psi) \quad \text{subject to} \quad g(\hat{\theta}, \psi) \leq 0, \;\; h(\hat{\theta}, \psi) = 0.$$

Linear Model



Objective: $\beta = \beta_{Y \sim D|X,Z,U}$

Ordinary Least Squares (OLS)

$$\beta_{Y \sim D|X,Z} = \frac{\mathrm{cov}(Y^{\perp X,Z}, D^{\perp X,Z})}{\mathrm{var}(D^{\perp X,Z})}$$

Unbiased, if at least one of the arrows is absent.

Two-Stage Least Squares (TSLS)

$$\beta_{\mathrm{TSLS}} = \frac{\mathrm{cov}(Y^{\perp X}, Z^{\perp X})}{\mathrm{cov}(D^{\perp X}, Z^{\perp X})}$$

Unbiased, if both of the arrows are absent.

Parameters θ and ψ : R-values

- Let Y be a random variable; X, W, Z be random vectors.
- Residual of Y after regressing out X denoted by $Y^{\perp X}$

R^2-value : $$R^2_{Y\sim X} := 1 - \frac{\mathrm{var}(Y^{\perp X})}{\mathrm{var}(Y)}$$

Partial R^2-value : $$R^2_{Y\sim X|Z} := \frac{R^2_{Y\sim Z+X} - R^2_{Y\sim Z}}{1 - R^2_{Y\sim Z}}$$

Partial R-value : $$R_{Y\sim X|Z} := \mathrm{corr}(Y^{\perp Z}, X^{\perp Z})$$

Partial f-value : $$f_{Y\sim X|Z} := \frac{R_{Y\sim X|Z}}{\sqrt{1 - R^2_{Y\sim X|Z}}}$$

Cinelli and Hazlett (2020)

R^2-calculus

(i) *Independence*: If $Y \perp\!\!\!\perp X$, then $R^2_{Y\sim X} = 0$.

(ii) *Independent additivity*: If $X \perp\!\!\!\perp W$, then $R^2_{Y\sim X+W} = R^2_{Y\sim X} + R^2_{Y\sim W}$.

(iii) *Decomposition of unexplained variance*: $1 - R^2_{Y\sim X+W} = (1 - R^2_{Y\sim X})(1 - R^2_{Y\sim W|X})$

(iv) *Recursion of partial correlation*:

$$R_{Y\sim X|W} = \frac{R_{Y\sim X} - R_{Y\sim W}R_{X\sim W}}{\sqrt{1 - R^2_{Y\sim W}}\sqrt{1 - R^2_{X\sim W}}}$$

(v) *Reduction of partial correlation*: If X and W is one-dimensional and $Y \perp\!\!\!\perp W$, then

$$R_{Y\sim X|W} = \frac{R_{Y\sim X}}{\sqrt{1 - R^2_{X\sim W}}}$$

(vi) *Three-variable restriction*: If X and W is one-dimensional, then

$$f_{Y\sim X|W}\sqrt{1 - R^2_{Y\sim W|X}} = f_{Y\sim X}\sqrt{1 - R^2_{X\sim W}} - R_{Y\sim W|X}R_{X\sim W}.$$

All statements are also true when Z is partialed out, i.e. add "$|\,Z$".

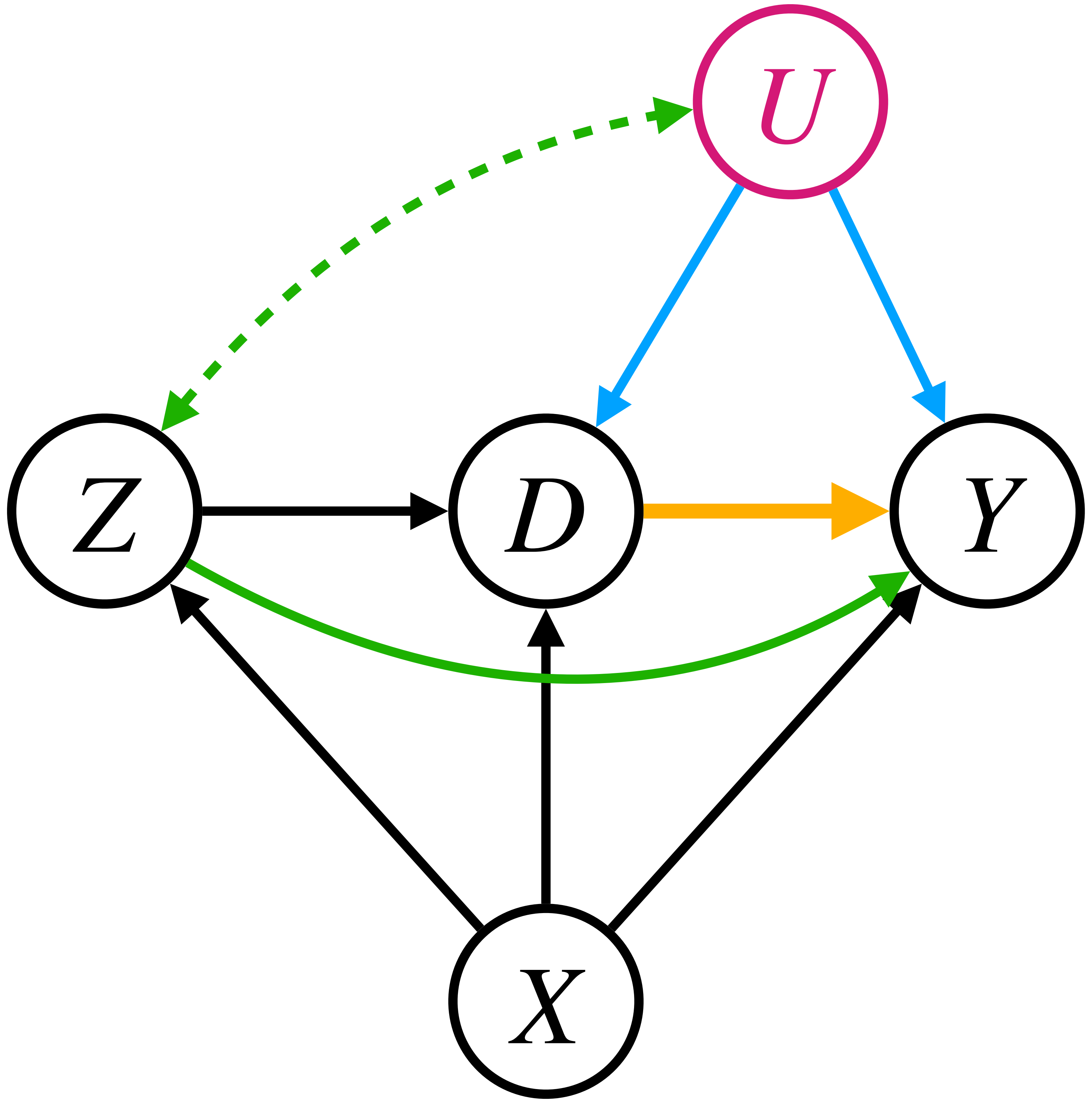
Anderson (1958)

Identifying objective β in terms of θ and ψ

Causal effect: $\beta = \beta_{Y\sim D|X,Z,U}$

$$\beta = \beta_{Y\sim D|X,Z} - R_{Y\sim U|X,Z,D}\, f_{D\sim U|X,Z} \frac{\mathrm{sd}(Y^{\perp X,Z,D})}{\mathrm{sd}(D^{\perp X,Z})}$$

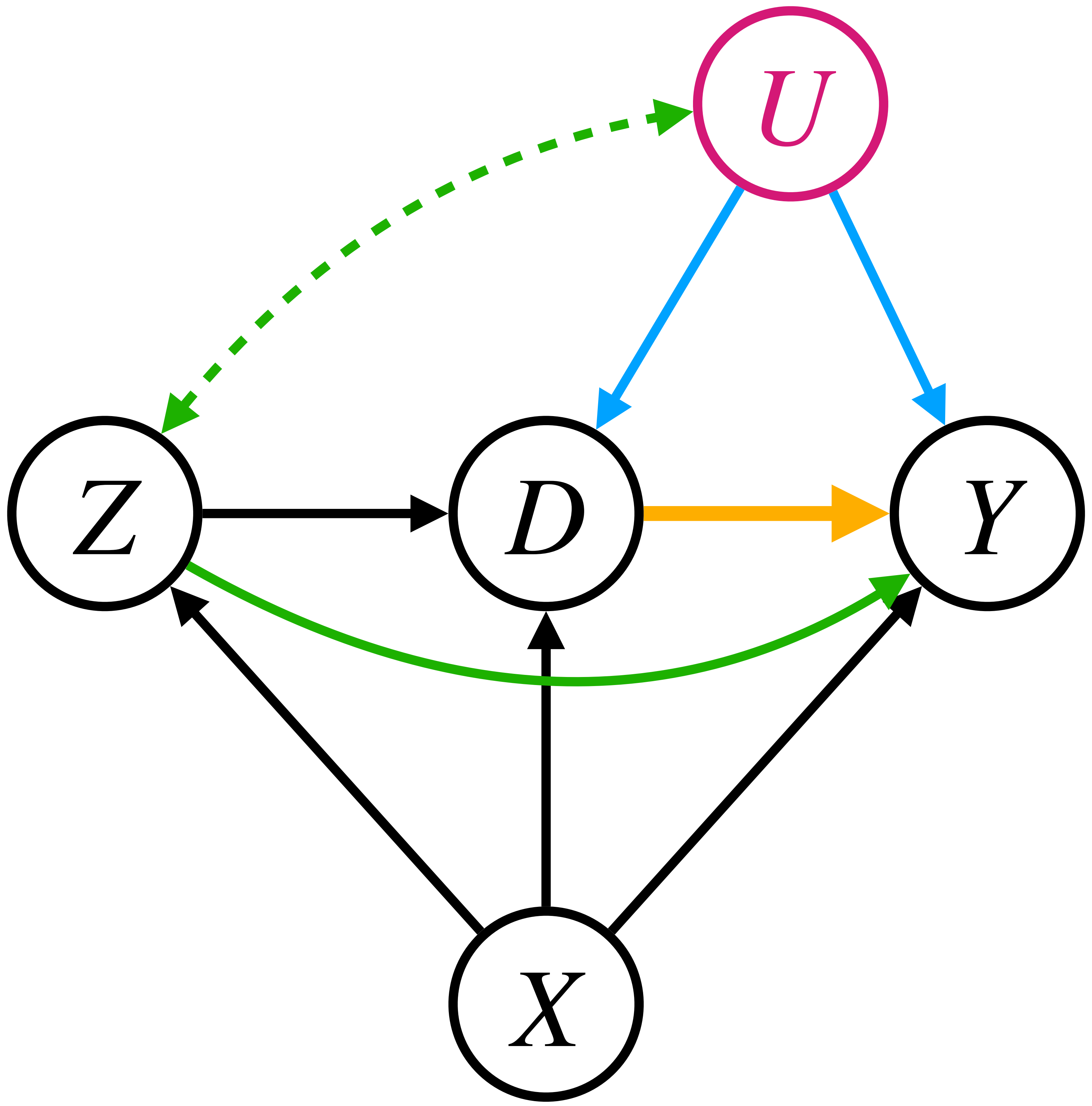
Sensitivity parameters ψ : $R_{Y\sim U|X,Z,D}$ and $R_{D\sim U|X,Z}$



Nagar (1959), Theil (1961)

Table of Bounds

$U \to D$	$R_{D\sim U\|X,Z} \in [B^l_{UD}, B^u_{UD}]$
	$R^2_{D\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UD}\, R^2_{D\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
$U \to Y$	$R_{Y\sim U\|X,D,Z} \in [B^l_{UY}, B^u_{UY}]$
	$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
	$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z,D} \leq b_{UY}\, R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	$R_{Y\sim Z\|X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Z\sim U\|\tilde{X},\dot{X}_{-j}} \leq b_{UZ}\, R^2_{Z\sim \dot{X}_j\|\tilde{X},\dot{X}_{-j}}$
$Z \to Y$	$R_{Y\sim Z\|X,U,D} \in [B^l_{ZY}, B^u_{ZY}]$
	$R^2_{Y\sim Z\|X,U,D} \leq b_{ZY}\, R^2_{Y\sim \dot{X}_j\|\tilde{X},\dot{X}_{-j},Z,U,D}$



The bounds can be combined in any way.

Cinelli and Hazlett (2020)

Constraints $g(\theta, \psi) \leq 0$

- Direct bounds, e.g. $R_{D\sim U|X,Z} \in [-0.2, 0.4]$, $R^2_{Y\sim U|X,Z,D} \leq 0.5$
- Assumption: $X = (\dot{X}, \tilde{X})$ such that $U \perp\!\!\!\perp \dot{X} \,|\, \tilde{X}, Z$
- Comparative bound, e.g. $R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z} \leq 2\, R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}$

The unmeasured confounder U can explain at most twice as much variance in Y than $\dot{X}_j$ does - conditional on $(\tilde{X}, \dot{X}_{-j}, Z)$.

Constraints $g(\theta, \psi) \leq 0$ and $h(\theta, \psi) = 0$

- $\min / \max \; \beta_{Y\sim D|X,Z} - R_{Y\sim U|X,Z,D}\, f_{D\sim U|X,Z} \dfrac{\mathrm{sd}(Y^{\perp X,Z,D})}{\mathrm{sd}(D^{\perp X,Z})}$ subject to (1), (2), (3)

- Bounds: $R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z} \leq 2\, R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}\,,\; R_{D\sim U|X,Z} \in [-0.2, 0.4]$ (1)

- $R^2_{Y\sim U|X,Z} \overset{\text{(v)}}{=} \dfrac{R^2_{Y\sim U|\tilde{X},\dot{X}_{-j},Z}}{1 - R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z}}$ (2)

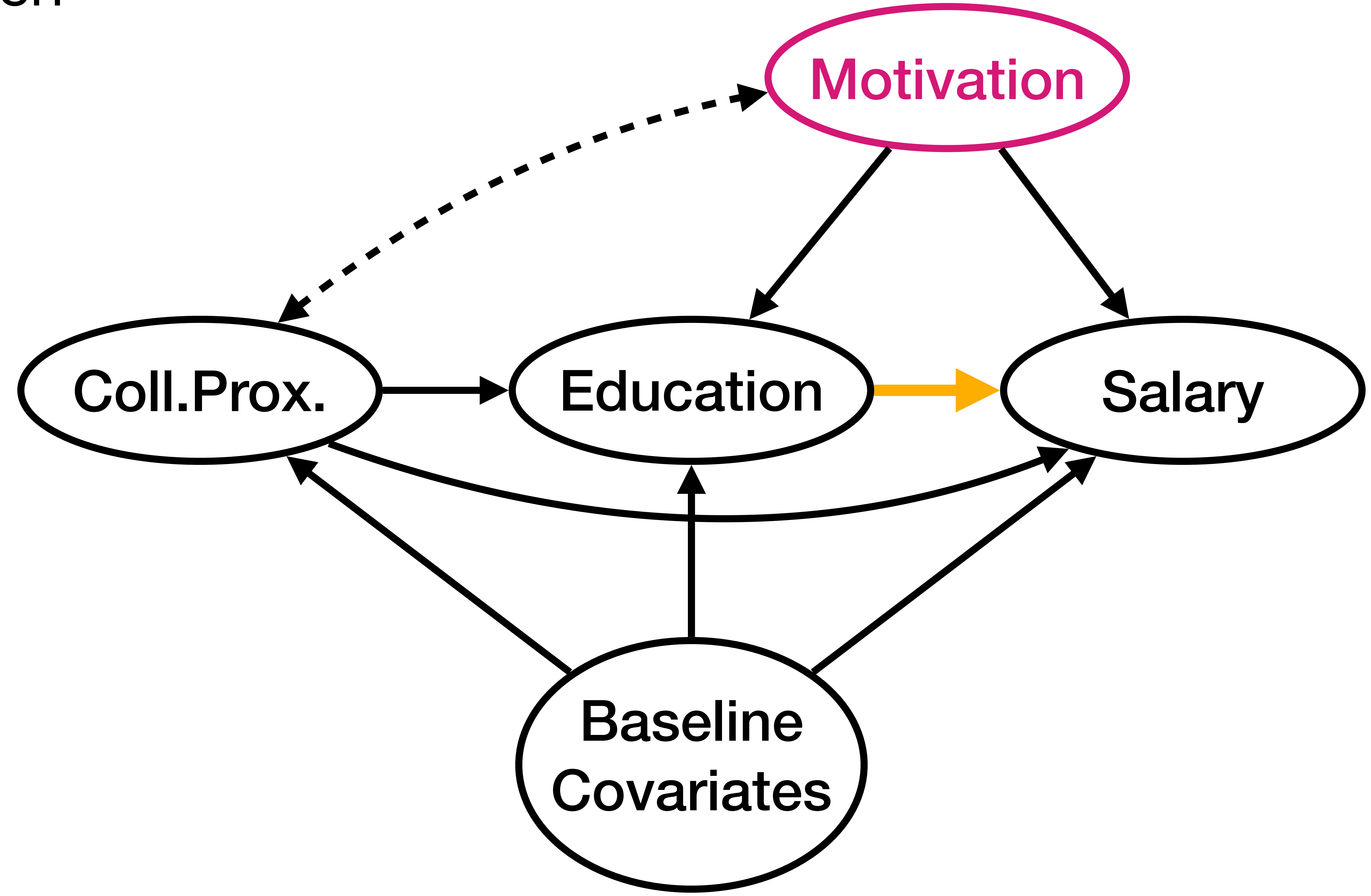
- $R_{Y\sim U|X,Z,D} \overset{\text{(iv)}}{=} \dfrac{R_{Y\sim U|X,Z} - R_{Y\sim D|X,Z}\, R_{D\sim U|X,Z}}{\sqrt{1 - R^2_{Y\sim D|X,Z}}\sqrt{1 - R^2_{D\sim U|X,Z}}}$ (3)

Optimization - Leveraging Monotonicity

- $a = R_{D\sim U|X,Z}$, $b = R_{Y\sim U|X,Z,D}$, $d = R_{Y\sim U|X,Z}$
- $\min / \max \ c_1 - b \dfrac{a}{\sqrt{1-a^2}} c_2, \quad$ s.t. $\quad a \in [-0.2, 0.4]$, $d^2 \leq c_3$, $b = \dfrac{d - c_4 a}{\sqrt{1 - c_4^2}\sqrt{1-a^2}}$
- Brute force grid search: create three-dimensional grid of (a, b, d)-values
- Adapted grid search:
  - For fixed a, b is monotone in d and the objective is monotone in b
  - One-dimensional grid of a-values
- $\mathcal{O}(m^3)$, but often $\mathcal{O}(m)$, where m is the number of points per grid dimension

Data Example

- National Longitudinal Survey of Young Men
- From 1966 until 1981
- 3010 individuals
- Y: log-earnings
- D: years of schooling
- X: years of labour force experience; indicators for living in the south, being black and living in a metropolitan area
- Z: growing up close to 4-year college



Card (1993)

Sensitivity Analysis

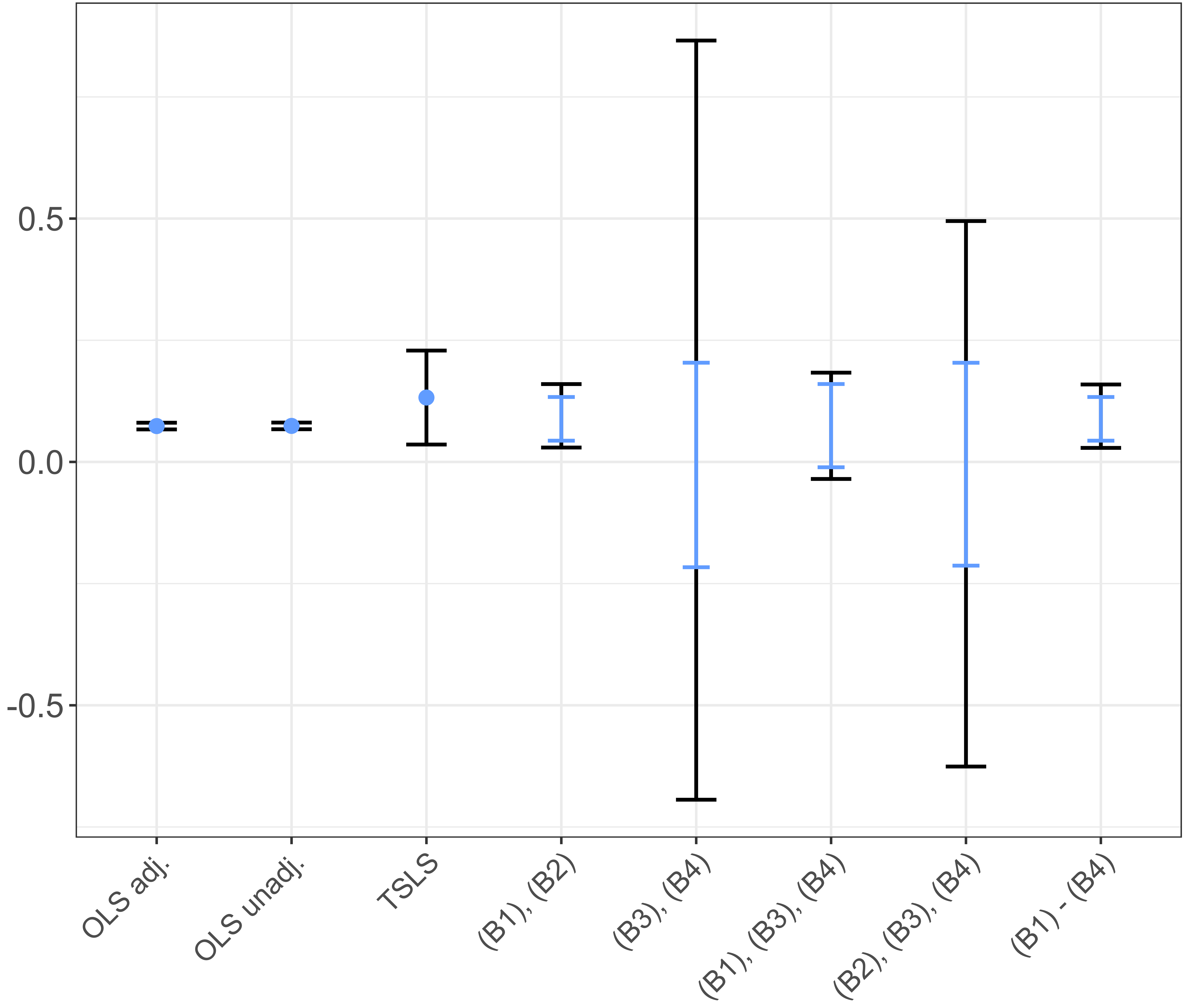
- Partition covariates X: $\dot{X}$ indicator for being black; $\tilde{X}$ remaining covariates
- Assumption: $U \perp\!\!\!\perp \dot{X} \mid \tilde{X}, Z$
- User specified bounds:

(B1) $R^2_{D \sim U | \tilde{X}, Z} \leq 4\, R^2_{D \sim \dot{X} | \tilde{X}, Z}$

(B2) $R^2_{Y \sim U | \tilde{X}, Z, D} \leq 5\, R^2_{Y \sim \dot{X} | \tilde{X}, Z, D}$

(B3) $R^2_{Z \sim U | \tilde{X}} \leq 0.5\, R^2_{Z \sim \dot{X} | \tilde{X}}$

(B4) $R^2_{Y \sim Z | X, U, D} \leq 0.1\, R^2_{Y \sim \dot{X} | \tilde{X}, Z, U, D}$



(B1): $U \to D$

(B2): $U \to Y$

(B3): $U \leftrightarrow Z$

(B4): $Z \to Y$

Things I did not talk about

- Extension of the R^2-calculus to Hilbert spaces
- Multiple unmeasured confounders
- Confidence statements: Sensitivity intervals via bootstrap
- Additional visualization tools

Conclusion

- New Framework: Sensitivity analysis as optimization problem
- Introduction of the R^2-calculus
- Sensitivity analysis in linear models with R-values (OLS and TSLS)
- Flexible and interpretable bounds
- Application on a dataset

Thanks for your attention!

arXiv: https://arxiv.org/abs/2301.00040

Website: https://tobias-freidling.onrender.com

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